**Power Series**
**Chapter 4**
**Section 23**

Let $f_n : D \to \mathbb{R}$ for $n = 0, 1, 2, 3, \ldots$ be functions. We study infinite sequences and series of functions.

For now we just do **power series** which are series of functions of the form

$$\sum_{n=p}^{\infty} a_n x^n$$

where usually $p = 0$ or 1.

(If all the $a_n$ were the same value we would have a geometric series.)

You may recall from Calculus that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x,$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x,$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n-1)!} = \sin x.$$

**Theorem.** Consider $\sum_{n=p}^{\infty} a_n x^n$. Let

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

Then $R$ is called the **radius of convergence** and

(i) the series converges if $|x| < R$,

(ii) the series diverges if $|x| > R$.

No conclusion can be drawn when $|x| = R$; one has to *check the end points.* Note that if $R = \infty$ the series converges for all $x$. The series always converges for $x = 0$.

The proof uses the Root Test. The version of the Root Test we did in class used only the limit of $\sqrt[n]{|a_n|}$ rather than the limit superior done in the textbook. The proof both cases is the same. The reason we need the stronger version we be clear when we do examples.

*Proof.* Case 1. Suppose $0 < R < \infty$. Then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n x^n|} = \frac{|x|}{R}.$$

For $|x| < R$ the series converges by the Root Test and for $|x| > R$ the series diverges by the Root Test.

Case 2. Suppose $R = 0$. Then for $x \neq 0$

$$\limsup_{n\to\infty} \sqrt[n]{|a_n x^n|} = \infty.$$

Hence Root Test implies the series diverges.

Case 3. Suppose $R = \infty$. Then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n x^n|} = 0 < 1$$

and the Root Test implies the series converges. □

In problems where the limit of the ratio $a_{n+1}/a_n$ exists it is equal to $\sqrt[n]{|a_n x^n|}$ by Corollary 12.3, so you can just use the Ratio Test. Example 6 in the textbook shows what can happen when this fails - we will discuss it in class.

**Example.** Find the radius of convergence of $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$.

*Proof.* Every odd term has $a_n = 0$. Thus, $\liminf \sqrt[n]{|a_n x^n|} = 0$. The $\limsup$ in the limit of the even terms.

$$\limsup_{n\to\infty} \sqrt[n]{|a_n x^n|} = \lim_{n\to\infty} \sqrt[2n]{|a_{2n} x^{2n}|} = |x| \lim_{n\to\infty} \frac{1}{\sqrt[2n]{(2n)!}}.$$

Now this is a subsequence of $\left(\frac{1}{\sqrt[n]{(n)!}}\right)$. This limit was Exercise 12.14a. It uses Corollary 12.3. Thus,

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{(n)!}} = \lim_{n\to\infty} \left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| = \lim_{n\to\infty} \left|\frac{1}{n+1}\right| = 0.$$

Thus, $R = \infty$. $\square$