Uniform Convergence of Functions
Section 24

Let (f_n) be a sequence of real valued functions all with domain $D \subset \mathbb{R}$. Let f be a real valued function with domain $D \subset \mathbb{R}$. We want to define what it means for (f_n) to converge to f. There are two different definitions that are used, **pointwise convergence** and **uniform convergence**.

It will turn out the uniform convergence implies pointwise convergence, but not the reverse. It will also turn out the the uniform limit of continuous functions is continuous, but this is not true for pointwise convergence.

Definition. We say (f_n) **converges pointwise** to f on D if

$$\forall x \in D \& \epsilon > 0, \exists\, N \text{ s.t. } n > N \implies |f_n(x) - f(x)| < \epsilon.$$

This is exactly the same as saying $f_n(x) \to f(x)$ for each $x \in D$. When this happens we write

$$\lim_{n\to\infty} f_n = f \text{ or } f_n \to f.$$

Definition. We say (f_n) **converges uniformly** to f on D if

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall\, x \in D,\, n > N \implies |f_n(x) - f(x)| < \epsilon.$$

When this happens we write

$$\lim_{n\to\infty} f_n = f \text{ or } f_n \to f.$$

Some books use $f_n \rightrightarrows f$.

Fact. It is immediate that uniform convergence implies pointwise convergence.

Example. Let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = x^n$. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{for} \quad x \in [0,1) \\ 1 & \text{for} \quad x = 1. \end{cases}$$

Then f_n convergences pointwise to f, but not uniformly.

Theorem. If $f_n \rightrightarrows f$ on D and the f_n functions are continuous on D, then f is continuous on D.

Proof. Let $x \in D$. We will prove f is continuous at x. Let $\epsilon > 0$. We observe that for any $n \in \mathbb{N}$

$$|f(x)-f(y)| = |f(x)-f_n(x)+f_n(x)-f_n(y)+f_n(y)-f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|.$$

Now $\exists\, n$ s.t. $|f(x)-f_n(x)| < \epsilon/3$ and $|f(y)-f_n(y)| < \epsilon/3$.

Fix such an n. Now $\exists \delta > 0$ s.t. $|x - y| < \delta$ implies $|f_n(x) - f_n(y)| < \epsilon/3$, since f_n is continuous (assuming $y \in D$).

Thus, $y \in D$ and $|x - y| < \delta$ imply $|f(x) - f(y)| < \epsilon$. Hence, f is continuous at x. Since x was an arbitrary point in D we have that f is continuous on D. Note: we are not claiming f in uniformly continuous on D. $\square$

There are many examples in the textbook. Be sure to study them.