Homework for Math 352
Spring 2020

Hwk Set 1. Due Monday, January 25. Do the following review problems from Calculus I & II (150/250), Linear Algebra (221) and Intro. to Proofs (302). Show all work.

1. $\lim_{t\to 2} \dfrac{t^2-4}{t-2}$
2. $\lim_{\theta\to 0} \dfrac{\sin 2\theta}{3\theta}$
3. $(\sec(e^x))'$
4. $\int x \sin x \, dx$
5. $\int x \sin x^2 \, dx$
6. $\int_{-1}^{1} \tan(x^3) \, dx$
7. Find the Taylor series centered about $x = 0$ of $\cos x^2$.
8. Find the interval of convergence of $\sum_{n=1}^{\infty} \dfrac{(2-x)^n}{n}$.
9. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, where A, B and C are sets.
10. Find the eigenvalues of $\begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$.

Hwk Set 2. Due Monday, February 1. Proofs are to be written in grammatically correct English sentences.

1. Prove that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ for all natural numbers n.
2. Prove that $\sum_{i=1}^{n} (2i-1) = n^2$ for all natural numbers n.
3. Prove $n^3 + 5n + 6$ is divisible by 3 for all natural numbers n.
4. Assume $r \neq 1$. Prove that
$$\sum_{k=0}^{n-1} r^k = \frac{r^n - 1}{r - 1},$$
for all natural numbers n.
5. Show that $3^{\frac{2}{7}}$ is not rational.
6. Show that $(2 + \sqrt{7})^{\frac{1}{5}}$ is not rational.

7. Graph $y = ||3x+5| - 2|$.
8. Prove that $\mathbb{Q}$ is countable.
9. Prove that $\mathbb{R}$ is not countable.
10. (Bonus!) Show that $\log_2 3$ is not rational.
11. (Bonus!) Prove that there are infinitely many prime natural numbers. Hint: Suppose the only prime numbers are $p_1, p_2, \ldots, p_n$. Think about $q = p_1 p_2 \cdots p_n + 1$.

Hwk Set 3. Due Monday, February 8.

1. Using only the properties of an ordered field prove the following.
   (a) $-a = -1 \cdot a$.
   (b) $(-a)(-b) = ab$.
   (c) $0 < 1$. Hint: Show that if $0 = 1$ then the "field" has only one member.
   (d) If $0 < a < b$, then $0 < b^{-1} < a^{-1}$.
   (e) If $a < b < 0$, then $b^{-1} < a^{-1} < 0$.
   (f) If $0 < a < b$, then $0 < a^2 < b^2$.
   (g) If $a < b < 0$, then $0 < b^2 < a^2$.
   (h) If $a < b$, then $a^3 < b^3$.
2. (Bonus!) (a) Let $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ be vectors in $\mathbb{R}^2$. Recall that for vectors the magnitude is given by $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$. Prove that the triangle inequality holds for vectors in $\mathbb{R}^2$, that is

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|.$$

   (b) Draw a picture illustrating this concept.
3. For each set below find the sup and inf; here $\pm\infty$ are allowed.
   (a) The domain of $\ln x$.
   (b) The range of e^x.
   (c) The range of $e^x \sin x$.
   (d) The range of $\frac{e^x+5}{3e^x+2}$.
   (e) The solution set of $\sin x = 0$.
   (f) The solution set of $\sin\left(\frac{1}{x}\right) = 0$.
   (g) The solution set of $\frac{x}{|x|} \sin^2\left(\frac{1}{x} + x\right) = \frac{1}{2}$.
   (h) Rational numbers r such that $r^2 < 5$.
   (i) Rational numbers r such that $r^3 < 5$.
   (j) $\{2 - \frac{1}{n} \mid n = 1, 2, 3, \ldots\}$.
4. (Bonus!) Let p and q be prime natural numbers. Prove that $\sqrt{p} + \sqrt{q}$ is not rational.

5. (Bonus!) Let $\mathbb{Z}_n$ denote the set $\{0, 1, 2, \ldots, n-1\}$. On $\mathbb{Z}_n$ we use addition and multiplication modulo n.

   (a) Set up addition and multiplication tables for $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$ and $\mathbb{Z}_5$.

   (b) Show that $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_5$ satisfy the field axioms: A1, A2, A3, A4, M1, M2, M3, M4, and DL. (They are not given an order relation.)

   (c) Show that $\mathbb{Z}_4$ satisfies A1, A2, A3, A4, M1, M2, M3, DL, but not M4.

   Note: It can be shown that $\mathbb{Z}_n$ is a field if and only if n is prime. You'll see this in MATH 419.

Hwk Set 4. Due Monday, February 15.

1. Determine for each series whether it converges to a real number, diverges to $\pm\infty$ or if the limit does not exist. If it converges to a real number find the limit. Prove all your claims using theorems in the textbook.
   (a) $a_n = \sqrt{\frac{n+1}{4n+7}}$.
   (b) $b_n = \frac{3n+5}{n^2+1}$.
   (c) $c_n = (-1)^n a_n$.
   (d) $d_n = (-1)^n b_n$.
   (e) $e_n = \sin(n\pi + \frac{1}{n})$.
   (f) $f_n = n - \sqrt{n^2 + n}$.

Hwk Set 5. Due Monday, February 22.

1. For each sequence below compute the first ten terms and plot these on the real n umber line. Find the inf, and sup of the underlying sets. Find the lim inf and lim sup of these sequences.
   (a) $\left\{\frac{(-1)^n}{n} + (-1)^{n+1} \,\middle|\, n = 1, 2, 3, \ldots\right\}$.
   (b) $\left\{(n \mod 3) + (-1)^n/n^2 \,\middle|\, n = 1, 2, 3, \ldots\right\}$
   (c) $\left\{(n \mod 3) - (-1)^n/n^2 \,\middle|\, n = 1, 2, 3, \ldots\right\}$
   (d) $\left\{\cos(n\pi/2) + (-1)^n/n \,\middle|\, n = 1, 2, 3, \ldots\right\}$
2. Suppose $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ and that $L \in \mathbb{R}$. Suppose that $a_n \to L$ and $c_n \to L$. Prove that $b_n \to L$. This is called the Squeeze Theorem.
3. Suppose $a_n \to L \in \mathbb{R}$. Prove that $|a_n| \to |L|$.
4. Suppose $a_n \leq b_n$ for all $n \in \mathbb{N}$. Prove that if $a_n \to \infty$, then $b_n \to \infty$.
5. Do Exercise 9.18 parts (b) & (c) on page 56. You did part (a) already. This is a standard test question.
6. (Bonus!) The textbook makes use of the binomial theorem several times. The proof is Exercise 1.12 parts (a), (b) & (c) on page 6. Do these!

Hwk Set 6. Due Monday, March 1. Due Monday, February 17. Do the following Exercises from your textbook. 14.2(a-f), 14.5(a), 14.6(a), 14.8, 15.4(a-d).

Hwk Set 7. Due Monday, March 8.

1. Let $f(x) = \begin{cases} x & \text{for} \quad x \in \mathbb{Q}, \\ 0 & \text{for} \quad x \notin \mathbb{Q}. \end{cases}$

Prove that f is continuous at $x = 0$, but is discontinuous everywhere else.

17.2 a,b.

18.5 a.

18.6.

Hwk Set 8. Due Monday, March 15.

19.2 a, b, c.

20.12 a, b, c.

Hwk Set 9. Due Monday, March 22.

23.2 a, b, c, d.

23.4 a, b, c.

24.1.

24.2 a, b, c.

24.16 a,b, c.

Hwk Set 10. Due Monday, March 29

1. Study Example 8 in Section 24 (pages 198-9). On a computer graph $f_n(x)$ for $x \in [0,1]$ for $n = 1, 2, 3, 5$, and 10. If you know how overlay the plots are print the result; if not print the individual graphs. Now, imagine you are teaching the class and write a paragraph explaining intuitively why $\lim_{n\to\infty} f_n(x) = 0$ for $x \in [0,1]$, but that the convergence is not uniform.

25.3(a) You can use the hint in the back of the book, but write out all the details.

25.6(a)

25.8

26.2 a, b, c.

26.3 a, b.

Hwk Set 11. Due Monday, April 5.

1. Prove that $(1-x) \displaystyle\sum_{k=m+1}^{n-1} x^k = x^{m+1}(1 - x^{n-m-1})$. (This is used in the proof of Abel's Theorem.)

2. Using only the defintion of the derivative and the Rules for Limits derive the derivatives of the two functions below.

a. $f(x) = \sqrt{3x+2}$ b. $g(x) = \frac{x}{x-1}$.

3. Using the Rules for Derivatives find the derivatives of the two functions below. You may assume the stadard results about derivatives of trigonometric, logarithmic and exponential functions.

a. $h(x) = \sqrt{e^{\cos 2x} + x}$. b. $p(x) = \sec^3(\ln 4x^2 + 1)$.

4. Let $f(x) = x + x^2 + x^3 + x^4$ on $[0, 1]$. Show that $f(x) = 1$ for one and only one value $x \in [0, 1]$. Hint: Use the IVT and the MVT.

28.4 a, b, c.

29.4

29.5

Hwk Set 12. Due Monday, April 12.

30.2 a, b, c, d.

30.6.

31.5 a, b.

X. Find the 6^{th} Taylor polynomial for each function below.

a. $\sec x$, about $c = 0$.

b. $e^{\sin x}$, about $c = 0$.

Y. Find the Taylor series of each function below. Determine its radius of convergence.

a. $f(x) = x/(1 + x^2)$, about $c = 0$.

b. $g(x) = x^3 + 2x$, about $c = 2$.

c. $h(x) = \sin^2 x$, about $x = 0$. Hint: use a trig identity.

Hwk Set 13. Due Monday, April 19.

1. Compute $\int_0^1 x^3 dx$ using the methods in Section 32.

2. In the proof of Lemma 32.2 (page 273) it was shown that $L(f, P) \leq L(f, Q)$ when $P \subset Q$. It was stated that the proof that $U(f, Q) \leq U(f, P)$ was similar. Write out the detail of this. (See also the lecture notes.)

3. [Bonus Question!]

A. Which of the following is a theorem in algebraic topology?
a. Furry nut theorem. b. Woolly plum theorem. c. Hairy ball theorem.

B. On which British game show did this question appear?
a. Anarchy in the U.K. b. The Chase. c. Harry and Megan's Not So Excellent Adventure.

C. Anarchy in the U.K. is obviously not a British game show. It is a song by which punk rock band?
a. The Buzzcocks. b. The Sex Pistols. c. The Dead Boys.

Hwk Set 14. Due Mondday, May 3

34.2 a (hint: L'Hospital's Rule + FTC).

34.8 a.

BONUS: 34.6.

36.4 a,b,c,d.

36.6 a,b.