Homework for Math 352
Spring 2022

Show all work. Proofs are to be written in grammatically correct English sentences.

Hwk Set 1. Due Wednesday, January 19.

1. $\lim_{t \to 2} \frac{t^2 - 4}{t - 2}$
2. $\lim_{\theta \to 0} \frac{\sin 2\theta}{3\theta}$
3. $(\sec(e^x))'$
4. $\int x \sin x \, dx$
5. $\int x \sin x^2 \, dx$
6. $\int_{-1}^{1} \tan(x^3) \, dx$
7. Find the Taylor series centered about $x = 0$ of $\cos x^2$.
8. Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{(2-x)^n}{n}$.
9. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, where A, B and C are sets.
10. Prove that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ for all natural numbers n.
11. Prove that $\sum_{i=1}^{n} (2i - 1) = n^2$ for all natural numbers n.
12. Prove $n^3 + 5n + 6$ is divisible by 3 for all natural numbers n.
13. Assume $r \neq 1$. Prove that
$$\sum_{k=0}^{n-1} r^k = \frac{r^n - 1}{r - 1},$$
for all natural numbers n.

Hwk Set 2. Due Monday, January 24.

1. Show that $3^{\frac{2}{7}}$ is not rational.
2. Show that $(2 + \sqrt{7})^{\frac{1}{5}}$ is not rational.
3. Show that $\log_2 3$ is not rational.
4. Graph the following, label all critical points and intercepts. (a) $y = ||3x + 5| - 2|$. (b) $y = |2x + 4| - |x - 7|$. Hint: This is a thinking problem, not a plug in the points problem.

5. Prove that $\mathbb{Q}$ is countable.
6. Prove that $\mathbb{R}$ is not countable.
7. Prove that there are infinitely many prime natural numbers. Hint: Suppose the only prime numbers are $p_1, p_2, \ldots, p_n$. Think about $q = p_1 p_2 \cdots p_n + 1$.
8. Using only the properties of an ordered field prove the following.
   (a) $-a = -1 \cdot a$.
   (b) $(-a)(-b) = ab$.
   (c) $0 < 1$. Hint: Show that if $0 = 1$ then the "field" has only one member.
   (d) If $0 < a < b$, then $0 < b^{-1} < a^{-1}$.
   (e) If $a < b < 0$, then $b^{-1} < a^{-1} < 0$.
   (f) If $0 < a < b$, then $0 < a^2 < b^2$.
9. (Bonus!) (a) Let $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ be vectors in $\mathbb{R}^2$. Recall that for vectors the magnitude is given by $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$. Prove that the triangle inequality holds for vectors in $\mathbb{R}^2$, that is

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|.$$

   (b) Draw a picture illustrating this concept.

Hwk Set 3. Due Monday, January 31.

1. For each set below find the sup and inf; here $\pm\infty$ are allowed.
   (a) The domain of $\ln x$.
   (b) The range of e^x.
   (c) The range of $e^x \sin x$.
   (d) The range of $\frac{e^x+5}{3e^x+2}$.
   (e) The solution set of $\sin x = 0$.
   (f) The solution set of $\sin\left(\frac{1}{x}\right) = 0$.
   (g) The solution set of $\frac{x}{|x|} \sin^2\left(\frac{1}{x} + x\right) = \frac{1}{2}$.
   (h) Rational numbers r such that $r^2 < 5$.
   (i) Rational numbers r such that $r^3 < 5$.
   (j) $\{2 - \frac{1}{n} \,|\, n = 1, 2, 3, \ldots\}$.
2. Suppose $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ and that $L \in \mathbb{R}$. Suppose that $a_n \to L$ and $c_n \to L$. Prove that $b_n \to L$. This is called the Squeeze Theorem.
3. Suppose $a_n \to L \in \mathbb{R}$. Prove that $|a_n| \to |L|$.
4. Suppose $a_n \leq b_n$ for all $n \in \mathbb{N}$. Prove that if $a_n \to \infty$, then $b_n \to \infty$.
5. Do Exercise 9.12, both parts, from page 55.
6. Do Exercise 9.18 parts (b) & (c) on page 56. You did part (a) already. This is a standard test question.

7. Do Exercise 10.6(a) from page 65.

Hwk Set 4. Due Monday, February 14.

1. For each sequence below compute the first ten terms and plot these on the real n umber line. Find the inf, and sup of the underlying sets. Find the lim inf and lim sup of these sequences.
   (a) $\left\{\frac{(-1)^n}{n} + (-1)^{n+1} \,\middle|\, n = 1, 2, 3, \ldots\right\}$.
   (b) $\left\{(n \mod 3) + (-1)^n/n^2 \,\middle|\, n = 1, 2, 3, \ldots\right\}$
   (c) $\left\{(n \mod 3) - (-1)^n/n^2 \,\middle|\, n = 1, 2, 3, \ldots\right\}$
   (d) $\left\{\cos(n\pi/2) + (-1)^n/n \,\middle|\, n = 1, 2, 3, \ldots\right\}$
2. Below you are given a formula for the n-th term of a **series**. Determine for each **series** whether it converges or diverges. Justify your answers.
   (a) $a_n = \sqrt{\frac{n+1}{4n+7}}$.
   (b) $b_n = \frac{3n+5}{n^2+1}$.
   (c) $c_n = (-1)^n a_n$.
   (d) $d_n = (-1)^n b_n$.
   (e) $e_n = \sin\left(n\pi + \frac{1}{n}\right)$. (You can assume standard properties of the sine and cosine functions.)
   (f) $f_n = n - \sqrt{n^2 + n}$.
3. Do the following Exercises from your textbook.
   14.2(a-f)
   14.5(a)
   14.6(a)
4. (Bonus!) The textbook makes use of the binomial theorem several times. The proof is Exercise 1.12 parts (a), (b) & (c) on page 6. Do these!

Hwk Set 5. Due Monday, February 21.

1. Let $f(x) = \begin{cases} x & \text{for} \quad x \in \mathbb{Q}, \\ 0 & \text{for} \quad x \notin \mathbb{Q}. \end{cases}$

   Prove that f is continuous at $x = 0$, but is discontinuous everywhere else.

2. Do the following from the textbook. Note: In this book, $\log x = \log_e x = \ln x$.

   14.7

   14.8

   15.3

   15.4(a-d)

   17.2(a-b)

   18.5(a)

   18.6

Hwk Set 6. Due Monday, February 28.

19.2 a, b, c.

19.4a

20.1

20.5

20.12 a, b, c.

20.18

Hwk Set 7. Due Monday, March 14. (But, but you can email it to me sooner to get feedback before Test 2 on March 14.)

23.2 a, b, c, d. [For b you can use l'Hopital's Rule.]

23.4 a, b, c.

24.1.

24.2 a, b, c.

24.16 a,b, c.

Extra Credit: Prove that $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if the inverse images of open sets are open.

Hwk Set 8. Due Monday, March 21.

1. Prove that $(1-x)\displaystyle\sum_{k=m+1}^{n-1} x^k = x^{m+1}(1 - x^{n-m-1})$. (This is used in the proof of Abel's Theorem.)

2. Using only the defintion of the derivative and the Rules for Limits derive the derivatives of the two functions below.

a. $f(x) = \sqrt{3x+2}$ b. $g(x) = \frac{x}{x-1}$.

3. Using the Rules for Derivatives find the derivatives of the two functions below. You may assume the stadard results about derivatives of trigonometric, logarithmic and exponential functions.

a. $h(x) = \sqrt{e^{\cos 2x} + x}$. b. $p(x) = \sec^3(\ln(4x^2 + 1))$.

Hwk Set 9. Due Monday, March 28.

1. Let $f(x) = x + x^2 + x^3 + x^4$ on $[0, 1]$. Show that $f(x) = 1$ for one and only one value $x \in [0, 1]$. Hint: Use the IVT and the MVT.

28.4 a, b, c.

29.4.

29.5.

30.1 a, b, c, d.

Hwk Set 10. Due Monday, April 4.

31.5 a, b.

A. Find the 5^{th} degree Taylor polynomial for each function below. (You may use a computer or calculator to find the derivatives. You can make a table with the results.)

a. $\sec x$, about $c = 0$.

b. $e^{\sin x}$, about $c = 0$.

B. Find the Taylor series of each function below. Determine its radius of convergence.

a. $f(x) = x/(1 + x^2)$, about $c = 0$.

b. $g(x) = x^3 + 2x$, about $c = 2$.

c. $h(x) = \sin^2 x$, about $c = 0$. Hint: use a trig identity.

Extra Credit. Let $f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$

This is Example 3 in your textbook, Section 31. We showed that $f^{(n)}(0) = 0$ for all $n \geq 0$. Hence, the Taylor series does not converge to $f(x)$ for $x > 0$. It must be that the conditions of Corollory 31.4 are not satified. Show this explicitly. That is, show that for any $C > 0$ and $\delta > 0$ there exists an $x \in (0, \delta)$ and an $n \in \mathbb{N}$ such that $|f^{(n)}(x)| > C$. Note: I have not done this problem and do not know how hard it is. You might graph $f^{(n)}(x)$ on a computer for a few values of n to see what is going on.

Hwk Set 11. Due Monday, April 11.

1. Compute $\int_0^1 x^3 dx$ using the methods in Section 32.

2. In the proof of Lemma 32.2 (page 273) it was shown that $L(f, P) \leq L(f, Q)$ when $P \subset Q$. It was stated that the proof that $U(f, Q) \leq U(f, P)$ was similar. Write out the detail of this. (See also the lecture notes.)

32.6

33.14

Hwk Set 12. Due Monday, April 18.

34.2 a (hint: L'Hospital's Rule + FTC).

34.8 a.

BONUS: 34.6.

Hwk Set 13. Due Wednesday, April 27.

Exercise 1 from the Arc Length lecture notes.

36.4 a,b,c.

36.6 a,b.

Hwk Set 14. Student Presentations.

Pick a problem or theorem or some combination to present. Your presenation should last 10-15 min. It will count 20 points toward. You can use the blackboard or the computer as you wish. Let me know your topic and what day will wish to present by Friday, April 22. I will resolve any conflicts and announce a schedule.