7.9 Using Undeterminal Coefficients

Ex Solve $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e^t \\ t \end{bmatrix}$

Sol First we solve the homogeneous problem $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

It is easy to check the eigenvalues are $3 \pm 2i$ with corresponding eigenvectors $\begin{bmatrix} 1 \\ \pm i \end{bmatrix}$. Thus, the general complex solution is

$$C_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{3t}(\cos(2t) + i\sin(2t)) + C_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{3t}(\cos(2t) - i\sin(2t))$$

To get the general real solution we find the real and imaginary parts of $\begin{bmatrix} 1 \\ i \end{bmatrix} e^{3t}(\cos(2t) + i\sin(2t))$

$$= e^{3t}\begin{bmatrix} \cos(2t) + i\sin(2t) \\ i\cos(2t) - \sin(2t) \end{bmatrix} = e^{3t}\begin{bmatrix} \cos(2t) \\ -\sin(2t \end{bmatrix} + i e^{3t}\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$$

Thus, the general real solution is

$$C_1 e^{3t}\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + C_2 e^{3t}\begin{bmatrix} \sin 2t \\ \cos(2t) \end{bmatrix}.$$

To find a particular solution let $P = \begin{bmatrix} a \\ b \end{bmatrix} e^t + \begin{bmatrix} c \\ d \end{bmatrix} t + \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} ae^t + ct + e \\ be^t + dt + f \end{bmatrix}$. We plug this into the original equation and then solve for a, b, c, d, e, f.

$$P' = \begin{bmatrix} ae^t + c \\ be^t + d \end{bmatrix}.$$ Thus

$$\begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} ae^t + ct + e \\ be^t + dt + f \end{bmatrix} + \begin{bmatrix} e^t \\ t \end{bmatrix} =$$

$$\begin{bmatrix} 3ae^t + 3ct + 3e + 2be^t + 2dt + 2f + e^t \\ -2ae^t - 2ct - 2e + 3be^t + 3dt + 3f + t \end{bmatrix} = \begin{bmatrix} ae^t + c \\ be^t + d \end{bmatrix}.$$

Hence,

① $3a + 2b + 1 = a$ ④ $-2a + 3b = b$

② $3c + 2d = 0$ ⑤ $-2c + 3d + 1 = 0$

③ $3e + 2f = c$ ⑥ $-2e + 3f = d$

①+④ $\Rightarrow$

$$\begin{aligned} 2a + 2b &= -1 \\ -2a + 2b &= 0 \Rightarrow a = b \\ \hline 4b &= -1 \Rightarrow b = -\tfrac{1}{4},\ a = -\tfrac{1}{4} \end{aligned}$$

②+⑤ $\Rightarrow$

$$\left.\begin{aligned} 3c + 2d &= 0 \\ -2c + 3d &= -1 \end{aligned}\right) \Rightarrow \begin{aligned} 6c + 4d &= 0 \\ -6c + 9d &= -3 \\ \hline 13d &= -3,\ d = \tfrac{-3}{13}. \end{aligned}$$

Then $c = \frac{-2d}{3} = \frac{2}{13}$

③+⑥ $\Rightarrow$

$$\left.\begin{aligned} 3e + 2f &= \tfrac{2}{13} \\ -2e + 3f &= \tfrac{-3}{13} \end{aligned}\right| \Rightarrow \begin{aligned} 6e + 4f &= \tfrac{4}{13} \\ -6e + 9f &= -\tfrac{9}{13} \\ \hline 13f &= \tfrac{-5}{13},\ f = \tfrac{-5}{13^2} = \tfrac{-5}{169} \end{aligned}$$

$$e = \frac{\frac{2}{13} - \frac{10}{169}}{3} = \frac{\frac{26-10}{169}}{3} = \frac{2}{169}$$

Thus, $P = \begin{bmatrix} -\frac{1}{4}e^t + \frac{2}{13}t + \frac{2}{169} \\ -\frac{1}{4}e^t + \frac{-3}{13}t - \frac{5}{169} \end{bmatrix}$

Thus, the general solution is

$$C_1 e^{3t}\begin{bmatrix}\cos 2t\\ -\sin 2t\end{bmatrix} + C_2 e^{3t}\begin{bmatrix}\sin 2t\\ \cos 2t\end{bmatrix} + \begin{bmatrix}-\frac{1}{4}e^{t} + \frac{2}{13}t + \frac{2}{169}\\ -\frac{1}{4}e^{t} - \frac{3}{13}t - \frac{5}{169}\end{bmatrix}.$$