Example Finding Two Invariant Planes

Here we study a 4×4 system, $\mathbf{v}'(t) = A\mathbf{v}(t)$, where the matrix A has four complex eigenvalues. They will of course be two pairs of complex conjugates. For each pair we will find an plane passing through the origin of $\mathbb{R}^4$ that is invariant. For one pair of eigenvalues the real parts are negative. In the corresponding invariant plane solution curves will spiral in toward the origin. For the other pair the real parts are positive, so for the other plane solutions curves spiral outward.

Let $A = \begin{bmatrix} 3 & 2 & 3 & 7 \\ -6 & 1 & -2 & -13 \\ 8 & -4 & 0 & 14 \\ -5 & 1 & 0 & -6 \end{bmatrix}$ and $\mathbf{v}(t) = \begin{bmatrix} w(t) \\ x(t) \\ y(t) \\ z(t) \end{bmatrix}$. We consider

$$\mathbf{v}'(t) = A\mathbf{v}(t).$$

Here are the eigenvalues for A and a choice of an eigenvector for each:

$$1+i,\ \mathbf{u}_1 = \begin{bmatrix} -1 \\ 2+i \\ -3-i \\ 1 \end{bmatrix};\ 1-i,\ \mathbf{u}_2 = \begin{bmatrix} -1 \\ 2-i \\ -3+i \\ 1 \end{bmatrix};\ -2+3i,\ \mathbf{u}_3 = \begin{bmatrix} -1-i \\ 3+i \\ -4 \\ 2 \end{bmatrix};$$

$$-2-3i,\ \mathbf{u}_4 = \begin{bmatrix} -1+i \\ 3-i \\ -4 \\ 2 \end{bmatrix}$$

Using these we can write down the general complex solution.

$$\mathbf{v}(t) = C_1\mathbf{u}_1 e^t(\cos t + i\sin t) + C_2\mathbf{u}_2 e^t(\cos t - i\sin t) + C_3\mathbf{u}_3 e^{-2t}(\cos 3t + i\sin 3t)$$

$$+\mathbf{u}_4 e^{-2t}(\cos 3t - i\sin 3t),$$

where C_1, C_2, C_3 and C_4 are complex constants.

With a bit of work you can show that the general real solution is

$$\mathbf{v}(t) = C_1 \begin{bmatrix} -\cos t \\ 2\cos t - \sin t \\ -3\cos t + \sin t \\ \cos t \end{bmatrix} e^t + C_2 \begin{bmatrix} -\sin t \\ \cos t + 2\sin t \\ -\cos t - 3\sin t \\ \sin t \end{bmatrix} e^t$$

$$+C_3 \begin{bmatrix} -\cos 3t + \sin 3t \\ 3\cos 3t - \sin 3t \\ -4\cos 3t \\ 2\sin 3t \end{bmatrix} e^{-2t} + C_4 \begin{bmatrix} -\cos 3t - \sin 3t \\ \cos 3t + 3\sin 3t \\ -4\sin 3t \\ 2\sin 3t \end{bmatrix} e^{-2t},$$

where C_1, C_2, C_3 and C_4 are real constants.

Now, there should be two invariant planes in $\mathbb{R}^4$, one where solutions spiral in toward the origin, and one where solutions spiral outward without bound. We can find basis vectors for these two subspaces by recombining the complex eigenvectors as follows. Let

$$\mathbf{r}_1 = \frac{\mathbf{u}_1 + \mathbf{u}_2}{2} = \begin{bmatrix} -1 \\ 2 \\ -3 \\ 1 \end{bmatrix}, \quad \mathbf{r}_2 = \frac{\mathbf{u}_1 - \mathbf{u}_2}{2i} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix},$$

$$\mathbf{r}_3 = \frac{\mathbf{u}_3 + \mathbf{u}_4}{2} = \begin{bmatrix} -1 \\ 3 \\ -4 \\ 2 \end{bmatrix}, \quad \mathbf{r}_4 = \frac{\mathbf{u}_3 - \mathbf{u}_4}{2i} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

Let

$$P_1 = \text{span}\{\mathbf{r}_1, \mathbf{r}_2\} = \{\alpha \mathbf{r}_1 + \beta \mathbf{r}_2 \,|\, \alpha, \beta \in \mathbb{R}\},$$

and

$$P_2 = \text{span}\{\mathbf{r}_3, \mathbf{r}_4\} = \{\delta \mathbf{r}_3 + \gamma \mathbf{r}_4 \,|\, \delta, \gamma \in \mathbb{R}\}.$$

Then P_1 is the invariant plane in $\mathbb{R}^4$ where solutions spiral out and P_2 is the invariant plane in $\mathbb{R}^4$ where solutions spiral in toward the origin.

It is challenging to visualize in four dimensions. (But see, *Flatland: A Romance of Many Dimensions*, by Edward Abbott, 1884, for help in developing your **mathematical imagination**.) However, we can abstract these two planes out of $\mathbb{R}^4$ and just see them as two dimensional planes.

Let's pick a point on P_1. That's a bit tricky. If we select four real numbers how do we know if they are coordinates of a point on P_1? We do not have an equation for P_1 like we do for planes in $\mathbb{R}^3$. (There is a no cross product in $\mathbb{R}^4$!). A point $(w, x, y, z) \in \mathbb{R}^4$ is on P_1 if and only if we can find real values of α and β such that

$$\alpha \begin{bmatrix} -1 \\ 2 \\ -3 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}.$$

In fact, we can generate points on P_1 by just picking values for α and β. Let's pick $\alpha = \beta = 1$. This gives the point $(-1, 3, -4, 1) \in P_1 \subset \mathbb{R}^4$. I used Maple to find the solution for this initial condition $\mathbf{v}(0) = \begin{bmatrix}-1 & 3 & -4 & 1\end{bmatrix}^T$. I got

$$\begin{bmatrix} w(t) \\ x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -\cos t - \sin t \\ 3\cos t + \sin t \\ -4\cos t - 2\sin t \\ \cos t + \sin t \end{bmatrix} e^t.$$

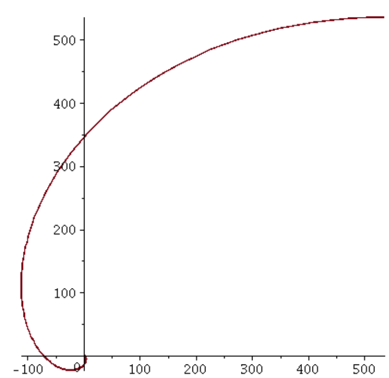
Here is the Maple command used.

```
> dsolve([diff(w(t),t) = 3*w(t)+2*x(t)+3*y(t)+7*z(t),
          diff(x(t),t) = -6*w(t)+x(t)-2*y(t)-13*z(t),
          diff(y(t),t) = 8*w(t)-4*x(t)+14*z(t),
          diff(z(t),t) = -5*w(t)+x(t)-6*z(t),
          w(0)=-1, x(0)=3, y(0)=-4, z(0)=1]);
```

Notice none of the terms in the solution involve e^{-2t}. We will graph it by solving for α and β and then graph the curve in the $\alpha\beta$-plane. Solving for $\alpha(t)$ and $\beta(t)$ gives,

$$\begin{aligned} \alpha(t) &= -w(t) = (\cos t + \sin t)e^t \\ \beta(t) &= x(t) - 2\alpha(t) = x(t) + 2w(t) = (\cos t - \sin t)e^t. \end{aligned}$$

Here is the graph of this solution curve in the $\alpha\beta$-plane.



Here is the Maple command that generated this plot.

```
> plot([(cos(t) + sin(t))*exp(t), (cos(t) - sin(t))*exp(t), t=0..2*Pi]);
```

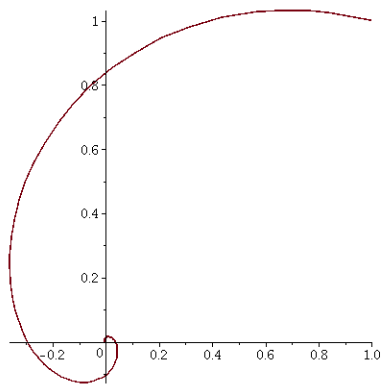
Let's play now with P_2. I'll pick the point on P_2 given by $\delta = 1$, $\gamma = 2$. This gives $(1, 1, -4, 2) \in P_2$. The solution curve, via Maple, is

$$\begin{bmatrix} w(t) \\ x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \cos 3t + 3\sin 3t \\ \cos 3t - 7\sin 3t \\ -4\cos 3t + 8\sin 3t \\ 2\cos 3t - 4\sin 3t \end{bmatrix} e^{-2t}.$$

Notice no terms involve e^t. Now we need to solve for δ and γ in terms of w, x. y and z.

$$\begin{aligned} \delta(t) &= z(t)/2 = (\cos t - 2\sin t)e^{-2t}. \\ \gamma(t) &= w(t) + \delta(t) = (2\cos 3t + \sin 3t)e^{-2t}. \end{aligned}$$

Below we plot this curve in the $\delta\gamma$-plane.



Here is the Maple command that generated this plot.

```
> plot([(cos(3*t) - 2*sin(3*t))*exp(-2*t), (2*cos(3*t) + sin(3*t))*exp(-2*t), t=0..2*Pi]);
```

If we pick any point in $\mathbb{R}^4$ as our initial starting point we can imagine (form an image of) what the system with do. Unless the point is exactly on P_2 the solution curve will move towards P_1 and as it get closer to P_1 it will spiral outward without bound. Only solutions starting on P_2 will converge to the origin. But, even in this case small computational errors could cause the apparent solution to leave P_2 and then it is off to infinity!