Example with six hyperbolic sectors, from Ordinary Differential Equations by Arrowsmith and Place, pg 92

<u>Ex</u>

$$x' = x(x-2y) = -2xy + x^2$$
$$y' = y(y-2x) = -2xy + y^2$$

Clearly $(0,0)$ is a critical point and it is easy to show it is the only one. Linearization does not provide much insight.

$$\text{Jacobian} = \begin{bmatrix} -2y+2x & -2x \\ -2y & -2x+2y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

We can gain some understanding by considering a few special cases.

Suppose $x=0$. Then $x'=0$ so the $y$-axis is invariant. Also $y' = y^2$. Thus, on the $y$-axis, except at $y=0$, the solution has increasing $y$, since $y'>0$.

We can solve for $y(t)$:

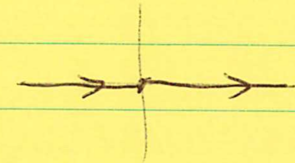
$$\frac{dy}{dt} = y^2 \iff \int y^{-2}\,dy = \int dt$$

$$\Rightarrow -\tfrac{1}{3}y^{-3} = t + c \Rightarrow y = \left(\frac{1}{c-3t}\right)^{1/3}.$$

If our init position is $y(0)=1$, then $c = -\frac{1}{3}$. Thus $y(t) \to \infty$ as $t \to 1$. As $t \to -\infty$, $y(t) \to 0$.

If $y(0) = -1$ you can do a similar analysis.

Suppose $y=0$. Then $y'=0$ and $x'=x^2$. So $x(t)$ is increasing for $t\neq 0$.

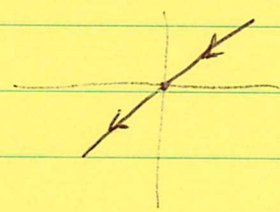


The behavior of solutions on the x-axis is similar to the y-axis case.

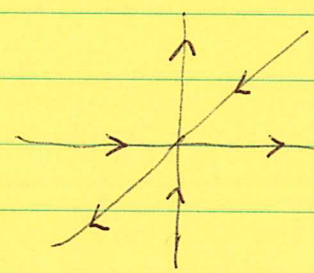
Next, suppose $x=y$. Does anything interesting happen?

$x' = -x^2 = -y^2$

$y' = -y^2 = -x^2$. Thus $x'=y'$, and so if we start on the line $x=y$, ~~we~~ solutions will stay on it.



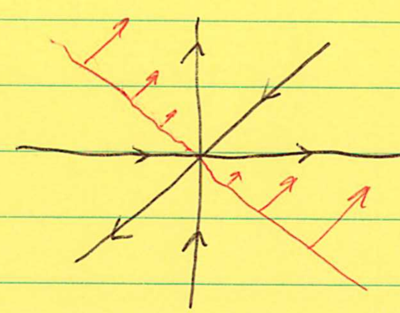
So far we have



What if $x=-y$? This line will not be invariant.

$x' = 3x^2$

$y' = 3y^2$ so $x'=y'$.

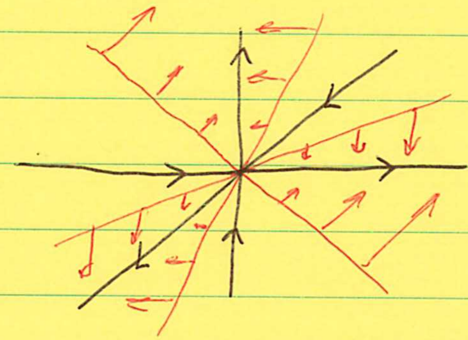


I looked at the lines $2x=y$ and $x=2y$.

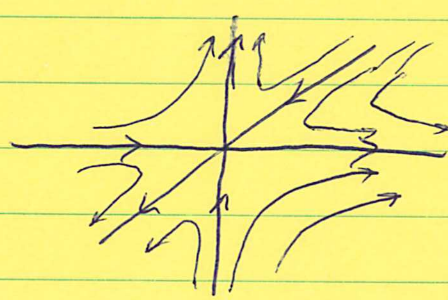
$2x=y \Rightarrow y'=0,\ x'=-3x^2$

$x=2y \Rightarrow x'=0,\ y'=-3y^2$

So, now we have



Here is a plausable phase portrait.



There seem to be 6 hyp. secters. Thus index $= 1-\frac{6}{2}=-2$.

Is this really correct? I was able to find implicit solution curves that confirm this picture. I'll also do computer plots.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2xy+y^2}{-2xy+x^2} = \frac{y}{x}\,\frac{y-2x}{x-2y}\,\frac{\frac{1}{x}}{\frac{1}{x}} = \frac{y}{x}\,\frac{\frac{y}{x}-2}{1-2\frac{y}{x}}.$$

Let $v = \frac{y}{x}$. Then $y = xv \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} = v + xv'$.

Thus

$$v + xv' = v\,\frac{v-2}{1-2v}$$

$$xv' = v\left(\frac{v-2}{1-2v} - \frac{1-2v}{1-2v}\right) = v\left(\frac{3v-3}{1-2v}\right)$$

$$\int \frac{1-2v}{3v(v-1)}\,dv = \int \frac{1}{x}\,dx = \ln|x| + c.$$

$$\shortparallel$$

$$-\frac{1}{3}\int \frac{1}{v} + \frac{1}{\cancel{v}-1}\,d\cancel{x} = \ln|x| + C$$

$$-\frac{1}{3}\left(\ln|v| + \ln|v-1|\right) = \ln|x| + C$$

$$\ln|v||v-1| = \ln|x|^{-3} + C$$

$$|v||(v-1)| = Cx^{-3}$$

$$\frac{y}{x}\left(\frac{y}{x} - 1\right) = Cx^{-3}$$

$$y(y-x) = \frac{C}{x} \qquad \boxed{xy(y-x) = C.}$$

Let's look at some initial values.

(1,1). $1(1-1) = \frac{c}{1} \Rightarrow c = 0. \Rightarrow y(y-x) = 0 \Rightarrow x = y$, which we already knew.

(-1,1) $1(1+1) = -c \Rightarrow c = -2.$

$$y(y-x) = \frac{-2}{x}$$

$$xy^2 - x^2y + 2 = 0$$

$$y = \frac{x^2 \pm \sqrt{x^4 - 8x}}{2x}$$

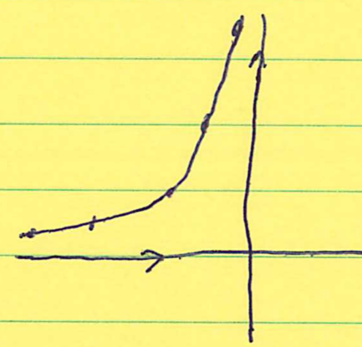
+ or − ? Let $x = -1$. $y = \frac{1 \pm \sqrt{9}}{-2}$

$= -2$ or $1$, hence use (−).

$$y = \frac{x^2 - \sqrt{x^4 - 8x}}{2x}$$

I plugged in a few numbers for x.

| x | y | |
|---|---|---|
| -1 | 1 | (given) |
| -2 | 0.4142 | |
| -3 | 0.1997 | |
| $-\frac{1}{2}$ | 1.766 | |
| $-\frac{1}{10}$ | 4.422 | |

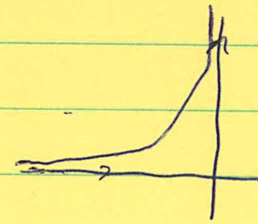


$$\lim_{x \to 0^-} \frac{x^2 - \sqrt{x^4 - 8x}}{2x} \cdot \frac{x^2 + \sqrt{x^4 - 8x}}{x^2 + \sqrt{x^4 - 8x}} = \lim_{x \to 0^-} \frac{x^4 - (x^4 - 8x)}{2x(x^2 + \sqrt{x^4 - 8x})}$$

$$= \lim_{x \to 0^-} \frac{4}{x^2 + \sqrt{x^4 - 8x}} = \infty.$$

$$\lim_{x\to-\infty} \frac{x^2-\sqrt{x^4-8x}}{2x} = \lim_{x\to-\infty} \frac{4}{x^2+\sqrt{x^4-8x}} = 0.$$

Thus, the solution curve through $(-1,1)$ ~~look~~ is "hyperbolic looking":



$(2,1)$. $\quad 1(1-2) = \frac{c}{2} \Rightarrow c = -2.$

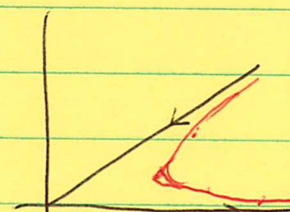
Again $y = \dfrac{x^2 \pm \sqrt{x^4-8x}}{2x}$, ~~but now we use ±, not~~ but now $x > 0$.

At $x=2$, $y=1$ is only solution. For ~~$x<2$~~ $0<x<2$, there are no real solutions.

Let $x=3$. $\quad y = 2.758$ and $0.2417$

Let $x=4$. $\quad y = 3.8708$ and $0.1292$

So, we should get the $(-)$ branch going to $0$ as $x\to\infty$, and the $(+)$ branch asymptotic to $x=y$ as $x\to\infty$.



$$\lim_{x\to\infty} \frac{x^2-\sqrt{x^4-8x}}{2x} = \lim_{x\to\infty} \frac{4}{x^2+\sqrt{x^4-8x}} = 0. \checkmark$$

$$\lim_{x\to\infty} \frac{y_+}{x} = \lim_{x\to\infty} \frac{x^2+\sqrt{x^4-8x}}{2x^2} = \lim_{x\to\infty} \frac{1+\sqrt{1-\frac{8}{x^3}}}{2} = \frac{1+\sqrt{1}}{2} = 1. \checkmark$$

You can study the other 4 sectors!

Maple 17

```
> with(DEtools):
> phaseportrait([(D(x))(t) = x(t)*(x(t)-2*y(t)), (D(y))(t) = y(t)*
  (y(t)-2*x(t))], [x(t), y(t)], t =-10..20, [[x(0) = 1, y(0) = 1],
  [x(0)=1,y(0)=0],[x(0)=0,y(0)=-1],[x(0)=0,y(0)=1],[x(0)=-1,y(0)=
  0],[x(0)=-1,y(0)=-1],[x(0)=-1/2,y(0)=1/2],[x(0)=1,y(0)=1/2],[x(0)
  =-1,y(0)=-1/2],[x(0)=1/2,y(0)=-1/2],[x(0)=1/2,y(0)=1],[x(0)=-1/2,
  y(0)=-1]],stepsize=0.1,color=black, linecolor=red,x=-2..2,y=-2.
  .2);
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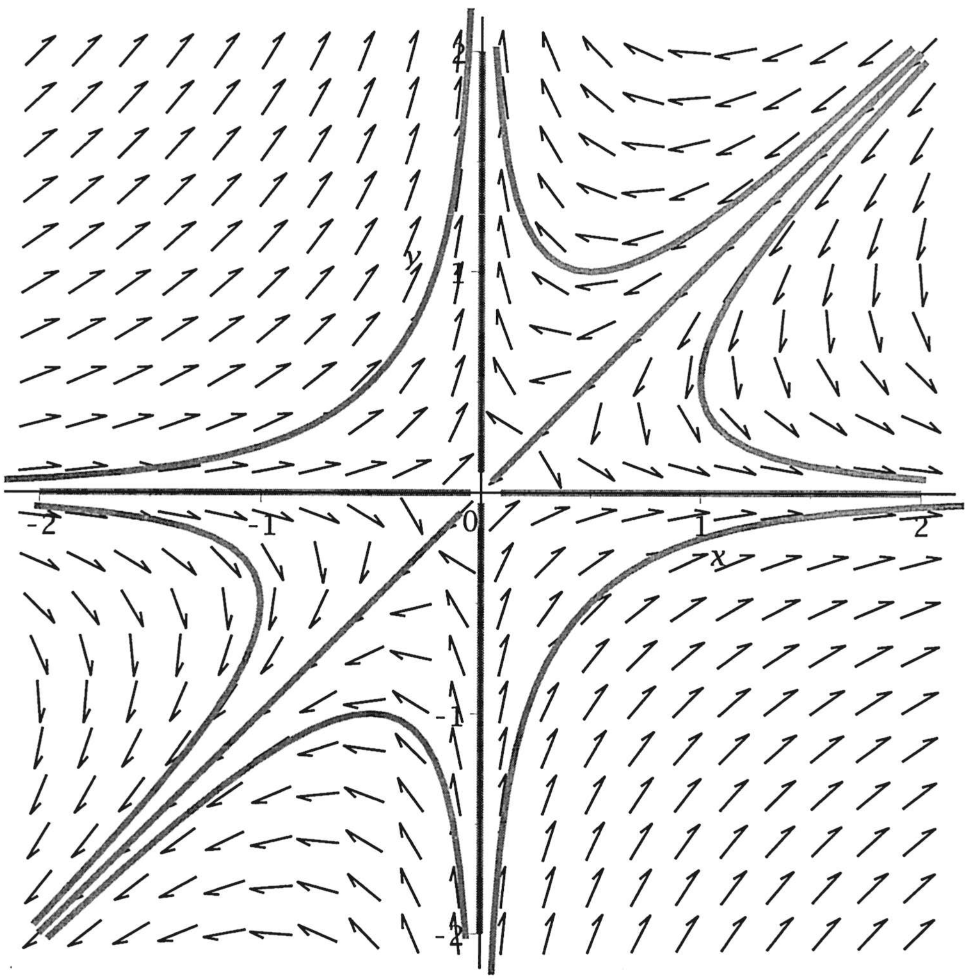


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>
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# Phase Plane

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**Phase Plane**

Equation 1 x(x-2y)

Equation 2 y(y-2x)

Interval on x -2,2

Interval on y -2,2

**Submit**

ctions below, then copy and paste into your HTML source.

xt/javascript" id="WolframAlphaSc

se only.

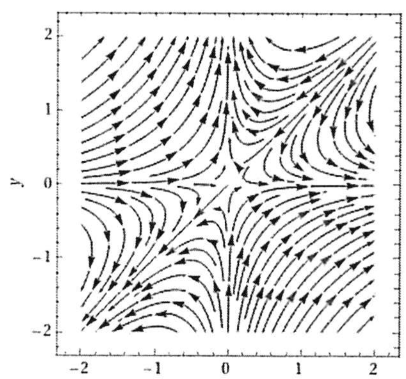
Output Width 575 px

Output Height 450 px

Input interpretation:

| stream plot | $(x(x-2y), y(-2x+y))$ | $x = -2$ to $2$ |
|---|---|---|
| | | $y = -2$ to $2$ |

Plot:



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Phase Planes

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