Examples of Degree Calculations.

Example 1. Let V(x,y) = (-y,x), and gamma(t) = (cos t, sin t) for t in [0, 2Pi]. Find degree of gamma wrt V.

In this example we let F(x,y) = -y = - sin t, and G(x,y) = x = cos t.

We shall integrate (G'F-GF')/(F^2+G^2) over [0,2Pi].

As luck would have it F^2 + G^2 = (-sin t)^2 + (cos t)^2 =1, and

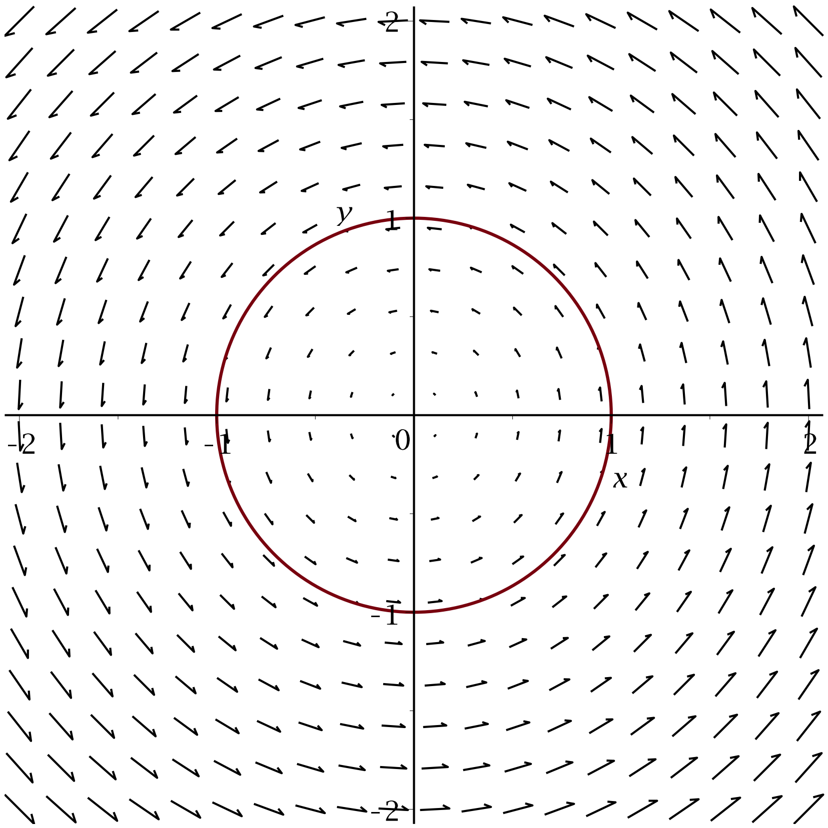
G' = (cos t)' = -sin t.
F'= (-sin t)' = -cos t.

Thus, G'F - GF' = ((-sin t)(-sin t) - (cos t)(-cos t) = 1.

Thus, the degree is, the integral of 1/1 from 0 to 2 Pi, divided by 2 Pi, which is 1.

Below is a plot of the vector field and the curve gamma.

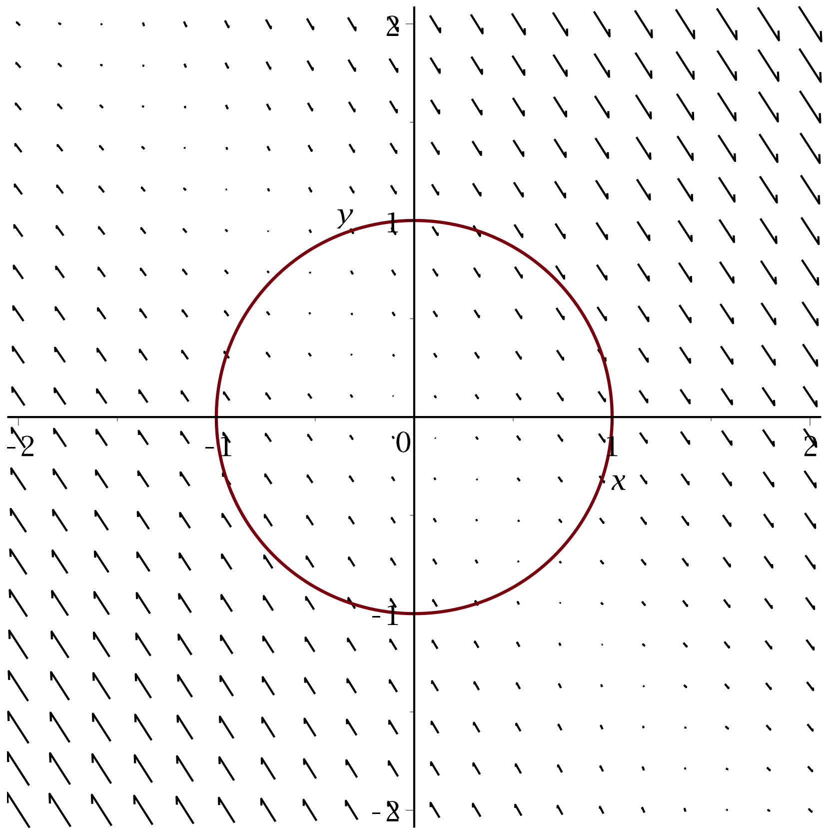
```
> vf:=fieldplot([-y,x],x=-2..2,y=-2..2):g:=plot([cos(t),sin(t),t=0..2*Pi]):display(vf,g);
```



Example 2. Let V(x,y) = (4x+3y, -6x-5y) and gamma(t) = unit circle = (cos t, sin t). Find the degree of gamma wrt V.

This time I did the plot first.

```
> vf:=fieldplot([4*x+3*y,-6*x-5*y],x=-2..2,y=-2..2):g:=plot([cos
  (t),sin(t),t=0..2*Pi]):display(vf,g);
```



This looks fishy, like there might zeros of V on gamma. But, if you check you will see that he eigenvalues of V are 1 and -2. So, (0,0) is a saddle, an isolated critical point.

Now F(t) = 4 cos t + 3 sin t, so F' = -4 sin t + 3 cos t.
And, G(t) = -6 cos t - 5 sin t, so G' = 6 sin t - 5 cos t.

We shall integrate (G'F-GF')/(F^2+G^2) over [0,2Pi]. and divide by 2 Pi. We do this with the computer next.

```
> int( ((6*sin(t)-5*cos(t))*(4*cos(t)+3*sin(t))-(-6*cos(t)-5*sin(t)
  )*(-4*sin(t)+3*cos(t)))/((4*cos(t)+3*sin(t))^2+(-6*cos(t)-5*sin
  (t))^2), t=0..2*Pi)/(2*Pi);
```

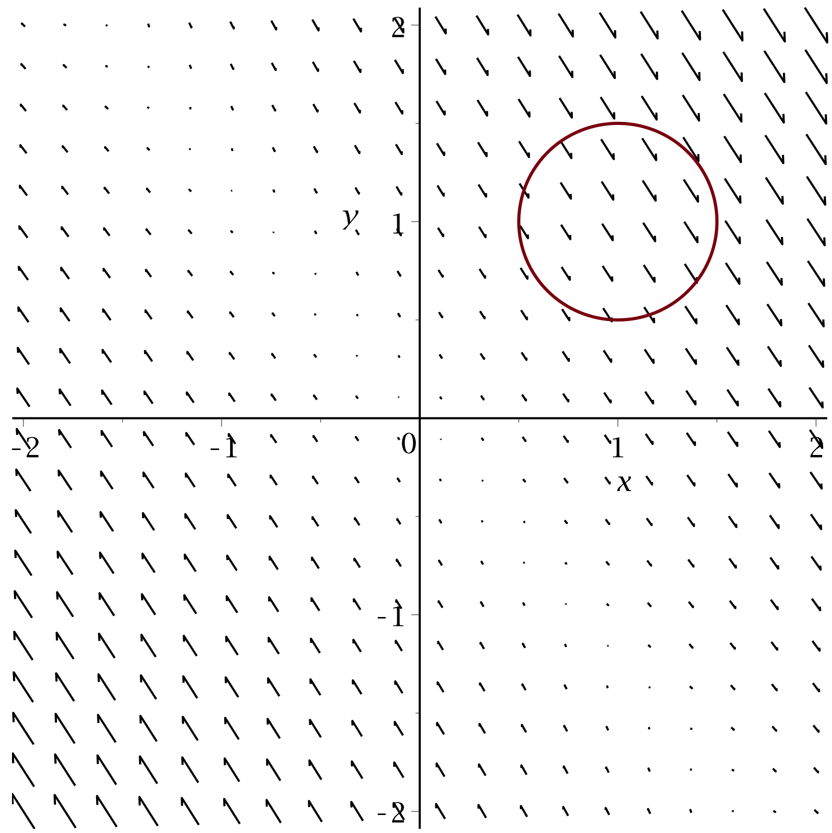
$$-1 \tag{1}$$

Thus, the degree is -1 as expected.

Example 3. Use the same vector field as in Example 2, but let gamma be the circle centered at (1,1) with radius 1/2.

Let gamma(t) = (cos(t)/2 + 1, sin(t)/2 + 1). A plot is below.

```
> vf:=fieldplot([4*x+3*y,-6*x-5*y],x=-2..2,y=-2..2):g:=plot([cos(t)/2 + 1,sin(t)/2 + 1,t=0..2*Pi]):display(vf,g);
```



Now F(t) = 4 (cos(t)/2 +1) + 3(sin(t)/2 + 1) = 7+ 2 cos t + 1.5 sin t, and G(t) = (-6(cos(t)/2 +1) - 5(sin(t)/2 +1) = -11 -3 cos t - 2.5 sin t.
Thus F'(t) = -2 sin t + 1.5 cos t and G'(t) = 3 sin t - 2.5 cos t.

We shall integrate (G'F-GF:)/(F^2+G^2) over [0,2Pi]. It would be hard to do this integral by hand, so I'll use the computer,

```
> int ( ((3*sin(t) - 2.5*cos(t))*(7 + 2*cos(t) + 1.5*sin(t)) - (-11 -3*cos(t)-2.5*sin(t))*(-2*sin(t)+1.5*cos(t)))/((7+2*cos(t)+1.5*sin(t))^2+(-11-3*cos(t)-2.5*sin(t))^2), t=0..2*Pi)/(2*Pi);
```

$$\frac{1}{2}\,\frac{1}{\pi}\left(\int_0^{2\pi}((3\sin(t)-2.5\cos(t))\,(7+2\cos(t)+1.5\sin(t))-(-11-3\cos(t)\right. \quad \textbf{(2)}$$

$$-2.5\sin(t))\,(-2\sin(t)+1.5\cos(t)))\Big/((7+2\cos(t)+1.5\sin(t))^2+($$

$$-11-3\cos(t)-2.5\sin(t))^2)\ dt)$$

This computer program, Maple 17, simply could not do this integral. So, I told it to do a nummerical estimate.

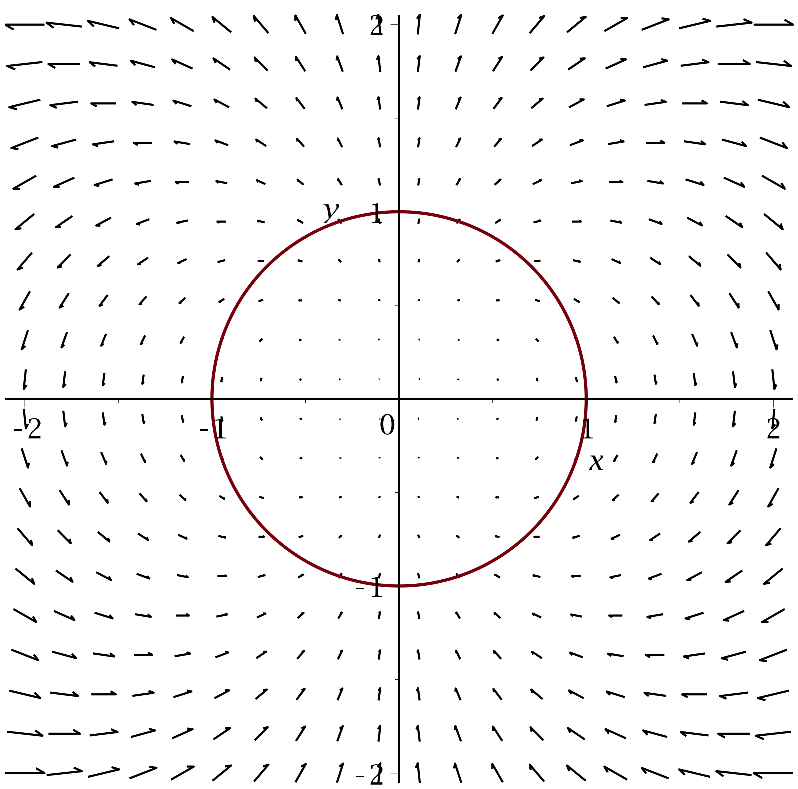
```
> evalf(%);
```

$$-3.582955588\ 10^{-16} \quad \text{(3)}$$

We will call this 0.

Example 4. Let V(x,y) = (2xy, y^2 - x^2). Let gamma(t) = (cos t, sin t). Find the dgree of gamma wrt V. The graph below shows we have a dipole field.

```
> vf:=fieldplot([2*x*y,y^2-x^2],x=-2..2,y=-2..2):g:=plot([cos(t),
  sin(t),t=0..2*Pi]):display(vf,g);
```



We have F(t) = 2 cos t sin t = sin 2t. So, F'(t) = 2 cos (2t).

We have G(t) = (sin t)^2 - (cos t)^2 = - cos 2t. So, G'(t) = 2 sin 2t.
Now F^2 + G^2 = (sin 2t)^2 + (-cos 2t)^2 = 1. Yea!
And, G'F - GF' = 2 (sin 2t)^2 - (-cos 2t)*2*(cos 2t) = 2. Yea!

Thus, the degree = int(2/1,t=0..2Pi) / 2Pi = 2.

Example 5. Let V(x,y) = (2x + y^2, x + y + xy). This was Example 1 is my lecture notes for Section 9.3. There it was shown that this vector field has three critical points, (0; 0), (-2; -2) and (-1/2, 1). We shall verify the Poincare Index Theroem by compting the index of each critical point and the find the degees of various simple closed curves. Note: I did the plot for this example on a separate worksheet. You may want to being looking at it as you read this.

First we find the index of (0,0). It is an unstable node, so we should get 1. A circle of radius 1, centered at (0,0) will not contain any other critical points, so we will use this curve to find the index of (0,0)..

F(t) = 2 cos t + (sin t)^2, so F'(t) = -2 sin t + 2 sin t cos t = -2 sin t + sin 2t.
G(t) = cos t + sin t + cos t sin t = cos t + sint + 0.5 sin 2t, This, G'(t) = -sin t + cos t + cos 2t.

Now we integrate (G'F - GF')/(F^2 + G^2) over [0,2Pi], and divide by 2 Pi.

```
> evalf(Int( ((-sin(t)+cos(t)+cos(2*t))*(2*cos(t)+sin(t)^2)-(cos(t)
  +sin(t)+sin(2*t)/2)*(-2*sin(t)+sin(2*t)))/((2*cos(t)+sin(t)^2)^2+
  (cos(t)+sin(t)+sin(2*t)/2)^2) ,t=0..2*Pi)/(2*Pi));
```

$$1.000000000 \quad (4)$$

The answer is 1.

Next we find the index of (-2,-2) using a circle of radius 1, cenetered at (-2,-2). Thus, gamma(t) = (-2+cos t, -2 + sin t). Recall it is a saddle, so we should get -1.

F(t) = -4 + 2 cos t + (-2 + sin t)^2 = 2 cos t - 4 sin t + (sin t)^2. Thus, F'(t) = -2 sin t - 4 cos t + 2 sin t cos t = -2 sin t - 4 cos t + sin 2t.

G(t) = -2+cos t + -2 + sin t + (-2+cos t)(-2 + sin t) = - cos t - sin t + (cos t)(sin t) = - cos t - sin t + 0.5 sin 2t. Thus, G'(t) = sin t - cos t + cos 2t.

Now we integrate (G'F - GF')/(F^2+G^2) over [0,2Pi] and divide by 2Pi.

```
> evalf(Int( ((sin(t)-cos(t)+cos(2*t))*(2*cos(t)-4*sin(t)+sin(t)^2)
  -(-cos(t)-sin(t)+sin(2*t)/2)*(-2*sin(t)-4*cos(t)+sin(2*t))) / (
  (2*cos(t)-4*sin(t)+sin(t)^2)^2+(-cos(t)-sin(t)+sin(2*t)/2)^2),t=
  0..2*Pi)/(2*Pi));
```

$$-1.000000000 \quad (5)$$

Yea!

Now we find the index of (-1/2, 1). We will use a circle of radius 1, centered at (-1/2, 1). Thus, gamma(t) = (-1/2 + cos t, 1 + sin t). This critical point was also a saddle.

F(t) = -1 + 2 cos t + (1 + sin t)^2 = 2 cos t + 2 sin t + (sin t)^2. Thus F'(t) = -2 sin t + 2 cos t + 2 sin t cos t = -2 sint + 2 cost + sin 2t.

G(t) = -1/2 + cos t + 1 + sin t + (-1/2 + cos t)(1 + sin t) = 0 + 2 cos t + 0.5 sin t + (cos t)(sin t) = 2 cos t + 0.5 sin t + 0.5 sin 2t.
thus, G'(t) = -2 sint + 0.5 cos t + cos 2t.

Now we integrate (G'F - GF')/(F^2+G^2) over [0,2Pi] and divide by 2Pi.

```
> evalf(Int( ((-2*sin(t)+cos(t)/2+cos(2*t))*(2*cos(t)+2*sin(t)+sin
  (t)^2)-(2*cos(t)+sin(t)/2+sin(2*t)/2)*(-2*sin(t)+2*cos(t)+sin(2*
  t)))/((2*cos(t)+2*sin(t)+sin(t)^2)^2+(2*cos(t)+sin(t)/2+sin(2*t)
  /2)^2), t=0..2*Pi)/(2*Pi));
```

$$-1.000000000 \quad (6)$$

Yea!

Now, if we choose a big circle centered at (0,0) that contains all three critical points, the Poincare Index Theroem says we should get 1 -1 -1 = -1.
Let gamma(t) = (3 cos t, 3 sin t) for t over [0, 2Pi].

F(t) = 6 cos t + 9 (sin t)^2. Thus, F'(t) = - 6 sin t+ 18 sin t cos t = -6 sin t + 9 sin 2t.

G(t) = 3 cos t + 3 sin t + 9 cos t sin t = 3 cos t + 3 sin t + 4.5 sin 2t. Thus, G'(t) = -3 sin t + 3 cos t + 9 cos 2t.

Now we integrate (G'F - GF')/(F^2+G^2) over [0,2Pi] and divide by 2Pi.

```
> evalf(Int( ( (-3*sin(t)+3*cos(t)+9*cos(2*t))*(6*cos(t)+9*sin(t)
  ^2)-(3*cos(t)+3*sin(t)+4.5*sin(2*t))*(-6*sin(t)+9*sin(2*t)) )/(
  (6*cos(t)+9*sin(t)^2)^2+(3*cos(t)+3*sin(t)+4.5*sin(2*t))^2 ), t=
  0..2*Pi)/(2*Pi));
```

$$-1.000000000 \quad (7)$$

If instead we used a circle of radius 2 centered at (0,0), only (0,0) and (-1/2, 1) will be inside it and the degree should be 1 + -1 = 0.

Let gamma(t) = (2 cos t, 2 sin t) for t over [0, 2Pi].

F(t) = 4 cos t + 4 (sin t)^2. F'(t) = - 4 sin t+ 8 sin t cos t = -4 sin t + 4 sin 2t..

G(t) = 2 cos t + 2 sin t + 4 cos t sin t = 2 cos t + 2 sin t + 2 sin 2t. Thus, G'(t) = -2 sin t + 2 cos t + 4 cos 2t.

Now we integrate (G'F - GF')/(F^2+G^2) over [0,2Pi] and divide by 2Pi.

```
> evalf(Int( ( (-2*sin(t)+2*cos(t)+4*cos(2*t))*(4*cos(t)+4*sin(t)
  ^2)-(2*cos(t)+2*sin(t)+2*sin(2*t))*(-4*sin(t)+4*sin(2*t)) )/( (4*
  cos(t)+4*sin(t)^2)^2+(2*cos(t)+2*sin(t)+2*sin(2*t))^2 ), t=0..2*
  Pi)/(2*Pi));
```

$$7.841613390\ 10^{-14} \quad \textbf{(8)}$$

We can take this to be zero.

On a separate Maple worksheet I'll show a phase plot for this system of equations with the circle overlaid.