Index Theory

Thm Brouwer's Fixed pt Thm. Let D be a closed disk in $\mathbb{R}^2$. Let $f: D \to D$ be cont. Then $\exists\, p \in D$ s.t. $f(p) = p$.

Such a point is called a <u>fixed pt</u>. Notice our thm fails for an annulus. Just rotate by $30°$. No fixed pts.

If $X \subseteq \mathbb{R}^2$ is s.t. every cont. map $f: X \to X$ must have a fixed pt, we say X has the <u>fixed point property</u>.

Two sets $X, Y \subseteq \mathbb{R}^2$ are regarded as <u>topologically equivalent</u> or <u>homeomorphic</u> if there is ~~a~~ map $f: X \to Y$ that is one-to-one
onto
cont.
and has cont. inverse.

The function f is called a homeomorphism.

Thm Let $X, Y \subseteq \mathbb{R}^2$ be homeomorphic. Then if X has the fixed pt prop. so does Y.

Pf Let $h: X \to Y$ be a homeo. Suppose $f: Y \to Y$ is cont. Let $g = h^{-1} \circ f \circ h : X \to X$. g is cont. Let $p \in X$ be a fixed pt, $g(p) = p$. Then

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "h"'] & X \\ Y \arrow[r, "f"'] & Y \arrow[u, "h^{-1}"'] \end{tikzcd}$$

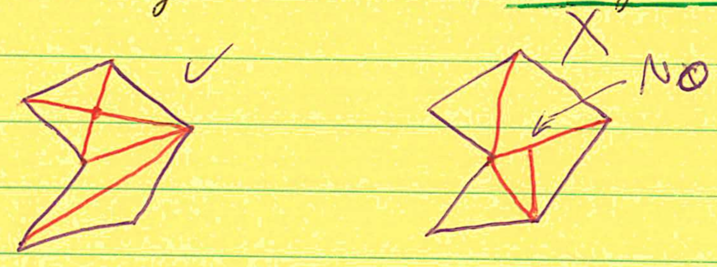
$$h^{-1}(f(h(p))) = p.$$

Thus, $f(h(p)) = h(p) \in Y$.

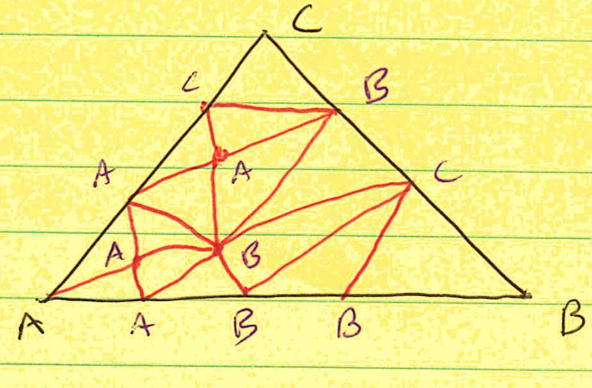
Thus $h(p)$ is a fixed pt. of f. $\square$

Thus, once we prove Brouwer's F.P. Th. we will actually have proved a thm about any set homeomorphic to a disk.

<u>Def</u> Let P be a polygon in $\mathbb{R}^2$. A division of P in to triangles s.t. each edge of each triangle is either in the boundary of P or is an edge of (only) one other triangle. is called a <u>triangulation</u> of P.



<u>Def</u> Let T be a triangle an then triangulate it further. A <u>Sperner labeling</u> is given by labeling each vertex with A, B or C as follows. Label the three vertices of the original T with A, B and C, using each letter once. Label the vertices of the smaller triangle's along the edges of T with the same letters as the end pts. Label the interior vertices any way you want.



Thm <u>Sperner's Lemma</u> At least one of the (small) triangles is labeled A,B,C, in any triangulated triangle with a Sperner ~~s~~ labeling.

Pf A triangle is called <u>"complete"</u> if it labels include each letter, A, B, C. Let x be the number of complete triangles. We will show x is odd. Hence x cannot be zero!

Let y be the numbers of triangles with labels A <u>and</u> B, and not C. That is y is the number of triangle labeled AAB or ABB. Such triangles have two edges labeled AB. Complete triangles have one such edge. All other triangles have no edges labeld AB. Thus, the total number of AB edge, counting triangle by triangle, is $2y + x$.

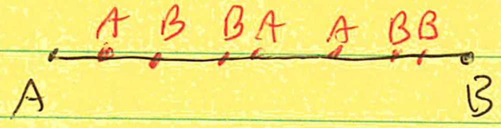
All possible labelings
AAA
AAB
ABB
BBB
AAC
ACC
CCC
BBC
BCC

Now, we counted each interior AB edge twice. Let z be the number of AB in the interior and w be the number of AB edges in the boundary. Then

$$2y + x = 2z + w.$$

I claim w is odd. This will force x to be odd.

Let's think about w. We need only consider the edge of T whose end pt were A and B. For example:



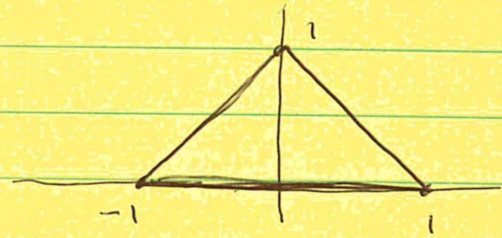
Let p be the number of AA segments. w is the number of AB (or BA) segments as stated. If we count the number of A vertices, edge by edge, we get $2p+w$. Again we counted each interior A twice. Let r be the number of internat A in this side of T. There is one A on the boundary. Thus

$$2p + w = 2r + 1.$$

Thus $w = 2r - 2p + 1$ is odd. Thus, x is odd. Thus x cannot be zero! $\blacksquare$

Now we are almost ready to prove B's Fixed Pt Thm. We will need the follow fact from topology: Let C be a closed, bounded set in $\mathbb{R}^2$. Let $(x_n)_{n=1}^{\infty}$ be an infinite seq. in C. Then $\exists$ a subseq $(x_{n_k})_{k=1}^{\infty}$ that converges to a point in C.

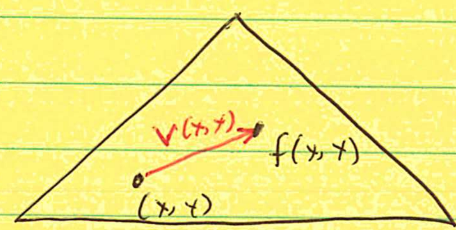
Proof of B's Fixed pt Thm. Consider the triangle below.



Let $f : T \to T$ be cont. Let $f(x,y) = (F(x,y), G(x,y))$. Let $V : T \to \mathbb{R}^2$ be

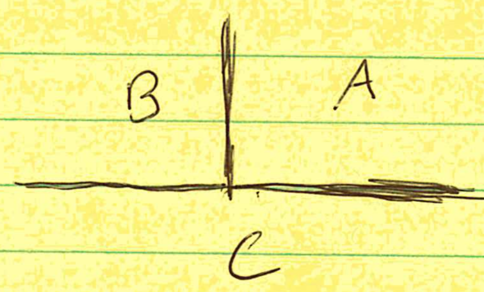
$$V(x,y) = (F(x,y) - x,\ G(x,y) - y)$$

and regard it as a vector based at (x,y) with the head at $(F(x,y), G(x,y))$



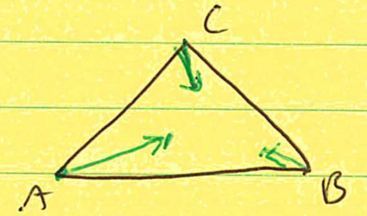
Consider a triangulation of T. We label the vertices A, B or C based on the vector at each vertex as follows. If p is a vertex let $V(p) = (u(p), w(p))$, but now regard it as based at the origin.

If $u(p) \geq 0$ and $w(p) \geq 0$ label p with A.
If $u(p) < 0$ and $w(p) \geq 0$ label p with B.
Otherwise use C.

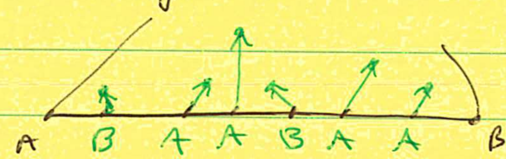


~~If $\exists$ a $p \in T$ w~~

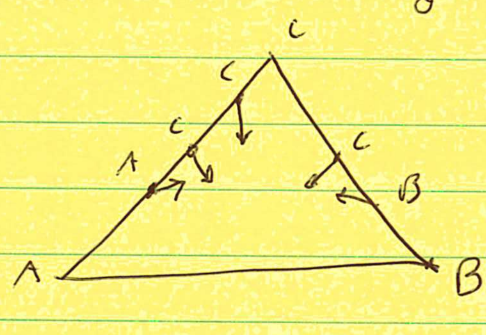
If any vertex has $V(p) = 0$, then it is a fixed pt and we are done. Assume this is not so. Then I claim the labeling is a Sperner labeling. First check the 3 vertices of T. Since the map is into T the labeling is clear. $(-1,0)$ is A, $(0,1)$ is B and $(0,1)$ is C.



Next look at the vertices along $(-1,1) \times 0$. All must be A or B.



Likewise for vertices along $\overline{AC}$ and $\overline{BC}$ of T.



The interior vertices can be labeled any way. Thus, we have a Sperner labeling. Select one of the ABC triangles. Let a_1, b_1 and c_1, be the coordinates of its vertices, a_1 for the vertex labeled A, etc.

We can insist all the edges in a triangulation be as small as we like. Let's say the edge in our triangulation where all ≤ 1. Then the points a_1, b_1 and c_1 are within 1 unit of each other.

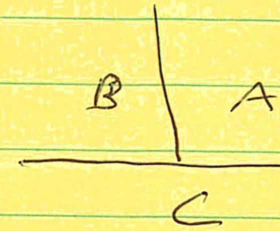
Now, select a new triangulation of T will all edges $\leq \frac{1}{2}$. Again, find an ABC triangle and let a_2, b_2, and c_2 be the coordinates of its vertizes.

Now, select a new triangulation of T will all edges $\leq \frac{1}{3}$. Again, find an ABC triangle and let a_3, b_3 and c_3 be the coordinates of its vertizes.

Keep doing this. We can generate three infinite sequences (a_n), (b_n) and (c_n) all inside T, a closed bounded set in $\mathbb{R}^2$. ~~Thus, each has a convergent subseq. Call the limits P_1, P_2, P_3. But wait. The dist'nt between any two of these three~~ Thus, (a_n) has a convergent subseq, (a_{n_k}). Let $a_{n_k} \to p \in T$. But (b_{n_k}) and (c_{n_k}) must also converge to p since $\text{dist}(a_{n_k}, b_{n_k}) \leq \frac{1}{n_k} \to 0$ and $\text{dist}(a_{n_k}, c_{n_k}) \leq \frac{1}{n_k} \to 0$.

Let's think about what $V(p)$ is. Since V is cont, the sequences $(V(a_{n_k}))$ $(V(b_{n_k}))$ and $(V(c_{n_k}))$ must all converge to $V(p)$.

Notice each $V(a_{n_k})$ points into the region A, and each $V(b_{n_k})$ points into the region B, and each $V(c_{n_k})$ points into the region C. Thus, their limits, $V(p)$, must be the one point in the boundary of all three regions, namely $(0,0)$. Thus $V(p) = (0,0)$. Thus $f(p)=p$ is fixed point!



$\square$

Corollary (For later use) Let $K \subset \mathbb{R}^2$ be closed and bounded. Suppose V is a continuous vector field on K that is never zero. Then $\exists\, \varepsilon > 0$ s.t. for any triangulation of K with edges of length $< \varepsilon$, there are no complete triangles.