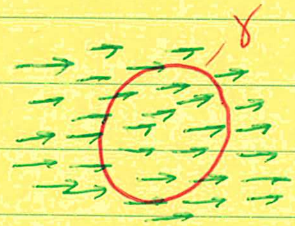
Degree of a simple closed curve with resp. to a vector field.

Let $V(x,y)$ be a continuous vector field on $\mathbb{R}^2$.
Let γ be a simple closed curve in $\mathbb{R}^2$ such that $V(x,y) \neq (0,0)$ when $(x,y) \in \gamma$.

We define the degree of γ with respect to V roughly as follows. Pick a point on γ and move along γ ccw. As you do so $V(x,y)$ will (likely) move but then must come back to where it was initially. Count how many times $V(x,y)$ make complete ($360°$, 2π) rotations as you go ccw around γ, counting each ccw revolution as $+1$ and each cw revolution as -1. The total is the degree of γ wrt V, denoted $d(\gamma)$.

Ex's



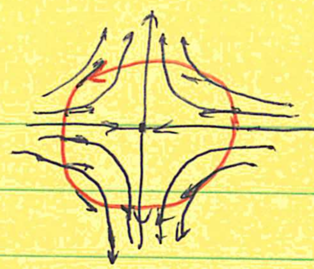
$d(\gamma) = 0$ for a constant v.f. for any γ

$d(\gamma) = 1$ for a repeller critical point for any curve γ containing the c. pt.

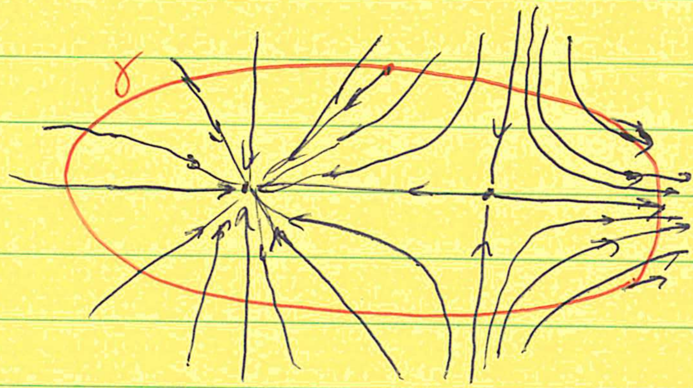
$d(\gamma) = 0$ if γ does not go around the c. pt.

More Examples

$d(\gamma) = 1$



$d(\gamma) = -1$

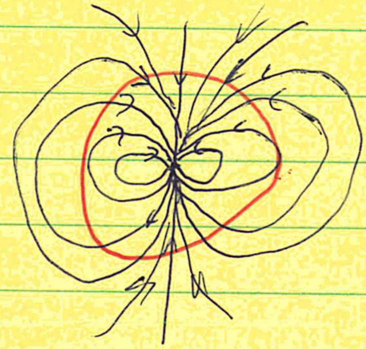


$d(\gamma) = 0$

c. pt. is center

γ is a solution curve (periodic)

$d(\gamma) = 1$

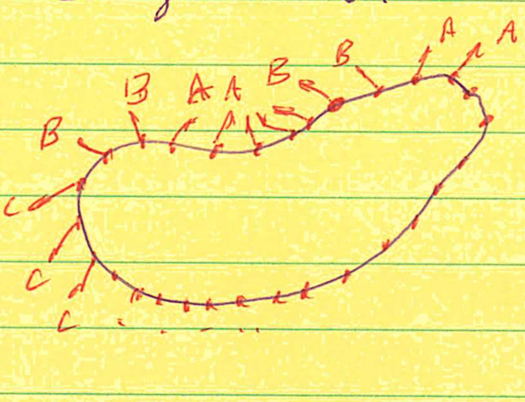


$d(\gamma) = 2$.

This is called a dipole.

We want to make this rough definition more formal. To do this we restate the corollary to the Bourwer Fixed Point Thm. Let $K \subset \mathbb{R}^2$ be compact (closed bounded). Suppose V is a cont. vector field that is never zero, $(0,0)$ on K. Then $\exists\ \varepsilon > 0$ s.t. if we pick any three points on K within ε of each other and label them each A, B or C using the scheme we used before, the triangle they form (we now allow degenerate triangles) will not be complete, ie will not have label A, B and C. The proof is the same argument by contradiction we did in the proof of the BFPT.

Thm Let V be a cont. v. f. on $\mathbb{R}^2$ and let γ be a SCC on which V is never zero. Then $\exists\ \varepsilon > 0$ s.t. if $P = \{p_i\}_{i=1}^{n}$ is a subdivision of γ s.t. consequtive p_i's are within ε of each other, then the index of P equals the degree of γ.



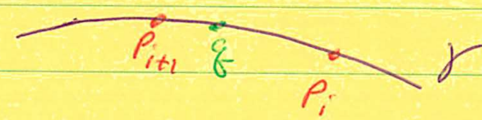
Pf Let $\varepsilon > 0$ be the number given by the corollary above.

Let P and Q be two subdivisions of γ where consequtive points are within ε of each other.

We need to show $I(P) = I(Q)$. We do this by showing

$$I(P) = I(P \cup Q) = I(Q)$$

Start with P and add a point q from Q. If q was already in P there is nothing to do. So assume $q \notin P$. Let q be in between p_i and p_{i+1}.



Label p_{i+1}, q, and p_i A, B, or C as usual. Since they are all within ε of each other the triangle cannot be complete. If it has all C's or only B's and C's or only A's and C's the contribution to the index was zero before we added q and zero after adding q, since the index only counts AB and BA edges. So, $I(P) = I(P \cup \{q\})$ in these cases.

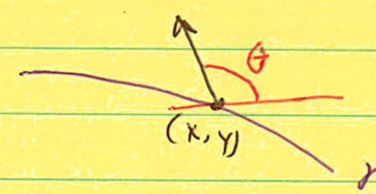
We list the remaining cases and note the effect on the index

	before	after
A A A	0	$0+0$
A B A	0	$1+-1=0$
A A B	1	$0+1=1$
A B B	1	$1+0=1$
B A A	-1	$-1+0=-1$
B B A	-1	$0+-1=-1$
B A B	0	$-1+1=0$
B B B	0	$0+0=0$

Hence, $I(P \cup \{q\}) = I(P)$. Since Q is finite we can repeat this and show $I(P \cup Q) = I(P)$, techincally, we are using the Principle of Mathematical Induction here. Likewise $I(Q) = I(P \cup Q)$.

Computing Degrees with Integration

Assume γ is piecewise smooth so integration along it is defined. Let $\theta(t)$ be the angle $V(x(t), y(t))$ makes with the $+x$ direction, where $\gamma(t) = (x(t), y(t))$.



Think of θ as a function of t. Let $V(x,y) = (F(x,y), G(x,y))$. Then

$$\theta(t) = \arctan\left(\frac{G}{F}\right).$$

Thus,

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{G}{F}\right)^2}\left(\frac{G}{F}\right)' = \frac{1}{1+\left(\frac{G}{F}\right)^2} \cdot \frac{G'F - GF'}{F^2}$$

$$= \frac{G'F - GF'}{F^2 + G^2}.$$

Thus, if $\gamma : [a,b] \to \mathbb{R}^2$, the total net change in θ is

$$\int_a^b \frac{G'F - GF'}{F^2+G^2}\, dt.$$

If we divide by 2π, we have the degree.

Ex Let $V(x,y) = (-y, x)$ and $\gamma(t) = (\cos t, \sin t)$. Find the degree of γ wrt V. $t \in [0, 2\pi]$

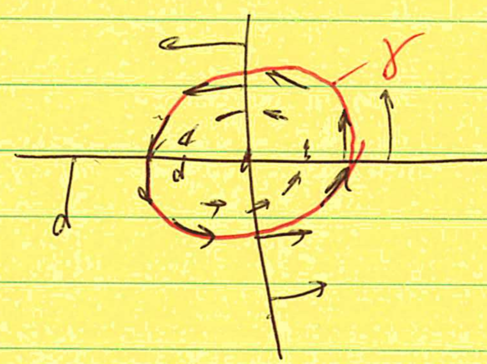
Sol $F(x,y) = -y = -\sin t$. Thus, $\dfrac{dF}{dt} = -\cos t$.

$G(x,y) = x = \cos t$. Thus $\dfrac{dG}{dt} = -\sin t$.

$$F^2 + G^2 = (-\cos t)^2 + (-\sin t)^2 = 1$$

$$G'F - GF' = (-\sin t)(-\sin t) - (\cos t)(-\cos t) = 1.$$

$$d(\gamma) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{1}\, dt = \frac{2\pi}{2\pi} = 1.$$



More examples are done with a computer on the website.