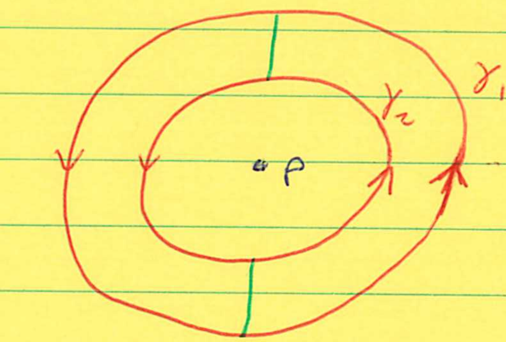
Index of an Isolated Critical Point

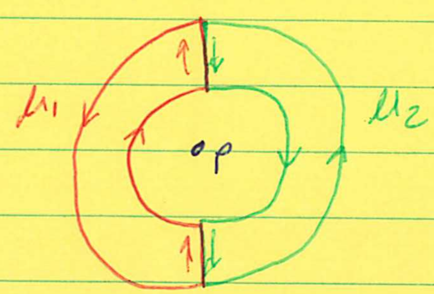
Thm The degree of a "small" circle containing an isolated critical point, and no other c. pts and that has no c. pts on it, is independent of its radius.

Def This justifies our defining the <u>index</u> of an isolated c. pt. as the degree of such a circle.

Pf Let γ_1 and γ_2 be two such circles centered on a c. pt. p. Draw two line segments from γ_1 to γ_2 along two radial lines as shown.



Now consider the two new simple closed curves μ_1 and μ_2 shown below.



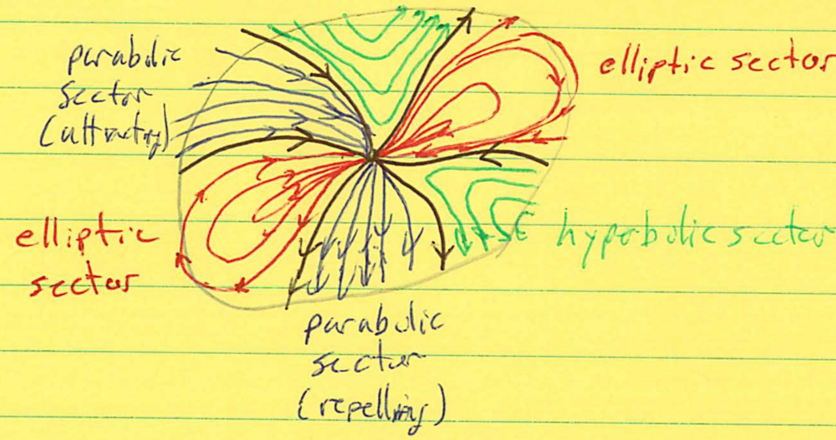
Orient them ccw. Then $d(\mu_1) = d(\mu_2) = 0$ since they enclose no c. pts. But,

$$d(\mu_1) + d(\mu_2) = d(\gamma_2) - d(\gamma_1)$$

implies, $d(\gamma_2) = d(\gamma_1)$. $\blacksquare$

A Theorem on Isolated Critical Points

Often the neighborhood of an isolated critical point can be divided into <u>sectors</u>. Below is an example



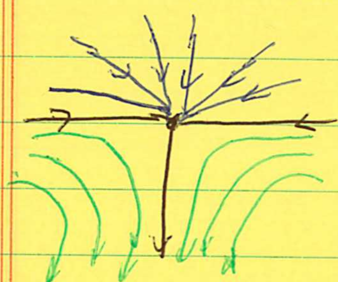
It has six sectors, two elliptic, two hyperbolic, and two parabolic. The solution curves the divide the sectors are called separatrixes.

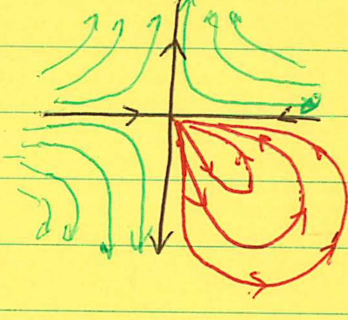
The linearizable systems we studied before had a single parabolic sector (attracting or repelling nodes) or (also improper nodes and spirals.) were saddles with four hyperbolic sectors, or ~~where~~ were centers that we are not considering here (they no sectors!).

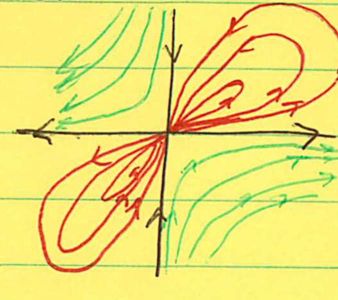
Although it is possible to have infinitely many sectors(!) we will not study these.

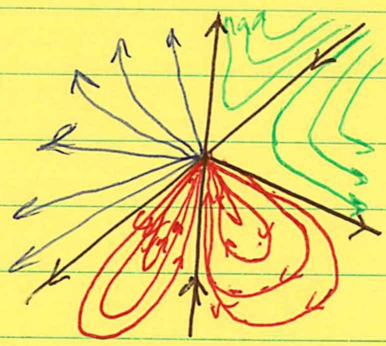
One could ask for a more precise definition of each type of sector, by we will just use our pictorial one above.

The index of the first example is 1. Take a few minutes to convince ourself of this. You might notice that the contributions of the ellipitic and hyperbolic sectors almost cancel out. Try the examples below.









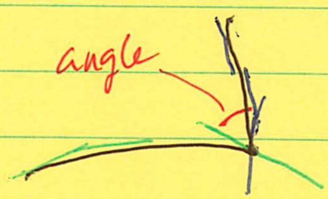
This is not an easy exercise. It may take you a while to get the hang of it. You may need to redraw them. The answers are 0, 0, 1, 1.

Let p be the number of parabolic sectors,
let e be the number of elliptic sectors.
Let h be the number of hyperbolic sectors.
Then the index is give by

$$I = 1 + \frac{e-h}{2}.$$

We will outline a proof of this.

First, each sector has an angle determined as follows. Give the two separtrixes that bound a given sector they have tangent lines at each point. As we move toward the critical point the tangent lines have "limits" lines and these determine an angle.



These angles will add up to 2π.

Let $\alpha_1, \alpha_2, \dots, \alpha_p$ be the angles for the parabolic sectors.
Let $\beta_1, \beta_2, \dots, \beta_e$ be the angles for the elliptic sectors.
Let $\gamma_1, \gamma_2, \dots \gamma_h$ be the angles for the hyperbolic sectors.

Each sector contributes to the index. By examining the examples, and making some of your own, you can see that if we let $a_1, \dots a_p$ be the contributions to the index from the parabolic sectors, $b_1 \dots b_e$, be the contributions from the elliptic sectors and $c_1, \dots c_h$ be the contributions from the hyperbolic sectors, then

$$a_i = \alpha_i \quad , i = 1, 2, 3, \dots p$$
$$b_i = \beta_i + \pi \quad , i = 1, \dots e$$
$$c_i = \gamma_i - \pi \quad , i = 1 \dots h.$$

$$2\pi \cdot I = a_1 + \dots + a_p + b_1 + \dots + b_e + c_1 + \dots + c_h = 2\pi + e\pi - h\pi$$

$$\Rightarrow \quad I = 1 + \frac{e-h}{2}.$$

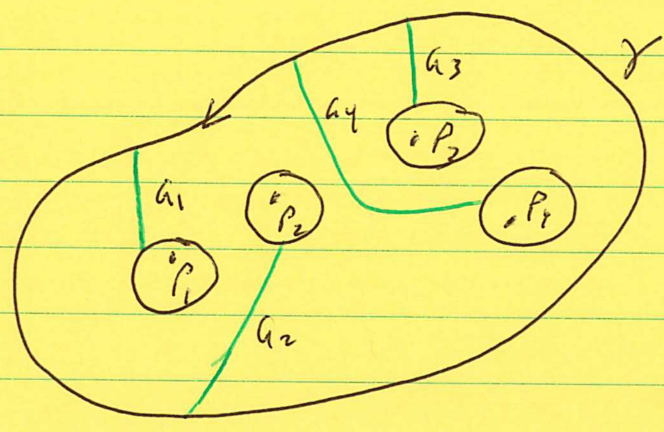
The Poincaré Index Theorem

Let $V(x,y)$ be a continuous vector field. Let γ be a simple closed curve on which V is never $(0,0)$. Let R be the region enclosed by γ and suppose V has finitely many (isolated) critical points inside R. Then the degree of γ wrt V is the sum of the indices of the critical points inside R,

$$d(\gamma) = I(p_1) + I(p_2) + \dots + I(p_n)$$

where $p_1, p_2, \dots p_n$ are the critical points of V inside R.

We will only illustraight the proof with an example. Around each critical point draw a small circle so that the circles are disjoint and do not meet γ.



Draw distinct arcs from γ to each circle so that the arcs are disjoint. Label them $a_1, \dots a_n$

Create a new curve γ' as follows. Go ccw around γ. Each time you meet an arc, a_i, take it to the circle around p_i. Then go cw around this circle and take a_i back to γ. Continue going ccw around γ. Do this unit you return to your starting point.

Now γ' is not a simple closed curve, since it self intersects on the arcs. But the proof that its degree is zero goes through as the region it bounds contains no critical points. Further its degree is

$$d(\gamma') = d(\gamma) - I(p_1) - I(p_2) - \cdots - I(p_n).$$

Thus, $d(\gamma) = I(p_1) + \cdots + I(p_n)$. $\square$