Thm Let $V(x,y) = (F(x,y), G(x,y))$ be a vector field in $\mathbb{R}^2$ where $F(x,y)$ and $G(x,y)$ have continuous first partial derivatives. Let $\gamma(t)$ be a piecewise smooth simple closed curve in $\mathbb{R}^2$. If $V(x,y)$ is never $(0,0)$ on or inside $\gamma(t)$, then the degree of $\gamma(t)$ with respect to $V(x,y)$ is zero.

We proved this before using combinatorical methods, remember index = content, with weaker assumptions. Here we will use calculus methods and Green's Theorem.

Pf Recall $d(\gamma) = \frac{1}{2\pi} \oint_\gamma \frac{G'F - GF'}{F^2+G^2} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\frac{dG}{dt} F - G \frac{dF}{dt}}{F^2+G^2}\, dt$

Now, $\frac{dG}{dt} = G_x \frac{dx}{dt} + G_y \frac{dy}{dt}$, by the 2-variable chain rule.

And $\frac{dF}{dt} = F_x \frac{dx}{dt} + F_y \frac{dy}{dt}$.

Thus,

$$d(\gamma) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(FG_x - F_yG)\frac{dx}{dt} + (FG_y - F_yG)\frac{dy}{dt}}{F^2+G^2}\, dt$$

$$= \frac{1}{2\pi} \int_\gamma \frac{FG_x - F_yG}{F^2+G^2}\, dx + \frac{FG_y - F_yG}{F^2+G^2}\, dy$$

$$= \frac{1}{2\pi} \iint_R \frac{\partial}{\partial x}\left(\frac{FG_y - F_yG}{F^2+G^2}\right) - \frac{\partial}{\partial y}\left(\frac{FG_x - F_yG}{F^2+G^2}\right) dA$$

$R \downarrow$ region inside γ

by Green's Thm, since F^2+G^2 is never zero.

I claim the integrand is zero.

$$\frac{\partial}{\partial x}\left(\frac{F G_y - F_y G}{F^2+G^2}\right) = \frac{(F G_y - F_y G)_x (F^2+G^2) - (F G_y - F_y G)(2 F F_x + 2 G G_x)}{(F^2+G^2)^2}$$

$$= \frac{(F_x G_y + F G_{yx} - F_{yx} G - F_y G_x)(F^2+G^2) + 2(F_y G - F G_y)(F F_x + G G_x)}{(F^2+G^2)^2}$$

$$= \Big[\cancel{F_x G_y F^2} + \cancel{G_{xx} F^3} - \cancel{F_{yx} G F^2} - \cancel{F_y G_x F^2}$$
$$+ \cancel{F_x G_y G^2} + \cancel{G_{yx} F G^2} - F_{yx} G^3 - \cancel{F_y G_x G^2}$$
$$+ 2\big(\cancel{F_x F_y F G} - \cancel{F_x G_y F^2} + \cancel{F_y G_x G^2} - G_x G_y F G \big) \Big] \Big/ \left[F^2+G^2\right]^2$$

Likewise, $\frac{\partial}{\partial y}\left(\frac{F G_x - F_y G}{F^2+G^2}\right)$

$$= \Big[\cancel{F_y G_x F^2} + \cancel{G_{xy} F^3} - \cancel{F_x G_y F^2} - \cancel{F_{xy} F^2 G}$$
$$+ \cancel{F_y G_y G^2} + \cancel{G_{xy} F G^2} - \cancel{F_x G_y G^2} - F_{xy} G^3$$
$$+ 2\big(\cancel{F_x F_y F G} - \cancel{F_y G_x F^2} + \cancel{F_y G_y G^2} - G_x G_y F G \big) \Big] \Big/ \left[F^2+G^2\right]^2.$$

Clearly, these cancel! (Recall $G_{xy} = G_{yx}$ and $F_{xy} = F_{yx}$.)

Thus, $d(\gamma) = \frac{1}{2\pi} \iint_R 0 \, dA = \frac{\text{area of } R}{2\pi} \cdot 0 = 0.$ $\boxed{/\!\!/}$