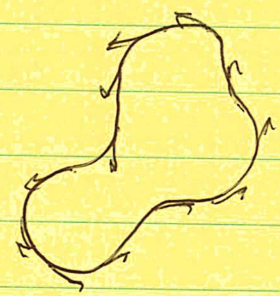
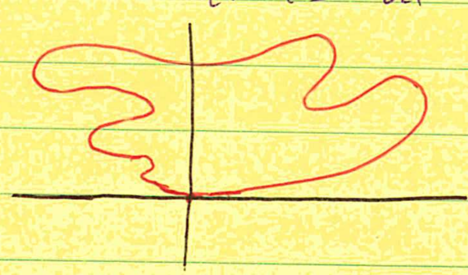
**Thm** (9.7.1 in your textbook) Let $V(x,y) = (F(x,y), G(x,y))$ be a vector field. Assume $F$ and $G$ have continuous first partial derivatives. Let $\gamma(t)$ be a period solution to $x' = F$, $y' = G$. Then $\gamma$ encloses at least one critical point.

In examples we've done visually the degree of a closed solution curve is one. If we accept this then the sum of the indices of the critical pts enclosed is also one. But if there were no c. pts. this sum would be zero, a contradiction.

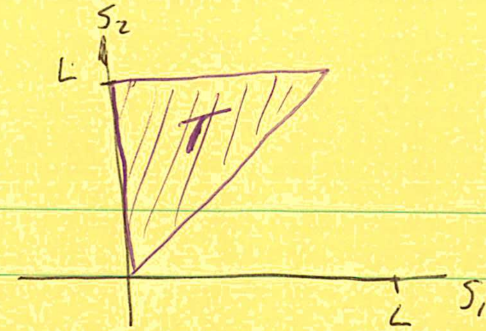


I'll present a formal proof that $d(\gamma)=1$, but it is a bit tricky.

First, we can translate the coordinate system so that $\gamma$ is in the upper half plane ($y \geq 0$) and so that $\gamma$ is tangent to the x-axis at $(0,0)$.



Next we parameterize $\gamma$ in terms of arc length, $s$, and so that $\gamma(0) = (0,0)$. Let $L$ be the total length,

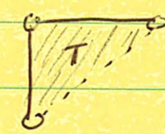



Let $T = \{(s_1, s_2) \mid 0 \le s_1 \le s_2 \le L\}$

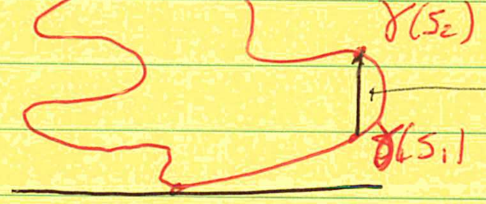
We are going to put a vector field $U(s_1, s_2)$ on $T$ that is related to the vectorfield $V(x,y)$ and the curve $\gamma(s)$.

For $s_1 \neq s_2$ and ~~not the 0 or L~~ not $(0, L)$, let $U(s_1, s_2) = \dfrac{\gamma(s_2) - \gamma(s_1)}{\|\gamma(s_2) - \gamma(s_1)\|}$.

This is well-defined and never zero on




The magnitude is always one. The direction is given by the secent line from $\gamma(s_1)$ to $\gamma(s_2)$,

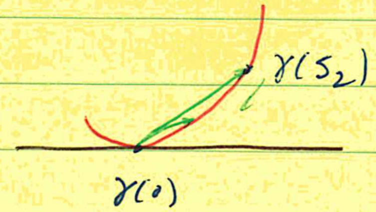



normal to get length one.
place at $(s_1, s_2)$ in $T$.

$U(s_1, s_2)$ is continuous since $\gamma(s)$ is and $\|\gamma(s_2) - \gamma(s_1)\| \neq 0$ for $s_1 \neq s_2$.

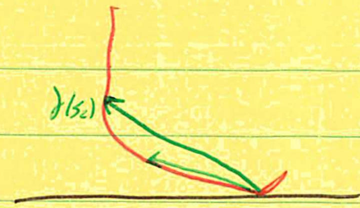
We will define $U(s_1, s_2)$ in the three corners using limits.

Let $U(0,0) = \lim\limits_{s_2 \to 0} U(0, s_2) = (1, 0)$.

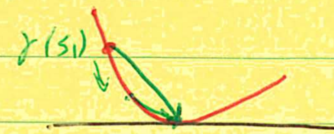



Limit will point is $+x$-axis direction with length 1. Thus $U(0,0) = (1,0)$.

Let $u(0, L) = \lim_{s_2 \to L} u(0, s_2) = (-1, 0)$




Let $u(L, L) = \lim_{s_1 \to L} u(s_1, L) = (1, 0)$.




Now for the remaining points, i.e. $s_1 = s_2$, (neither 0 or L) we let

$$u(s_1, s_1) = \frac{V(\gamma(s_1))}{\|V(\gamma(s_1))\|}$$

This makes $u$ continuous since the unit tangent vector is the limit of the normalized secant vectors.

Let $\mu$ be the boundary of T going ccw. Since $u(s_1, s_2)$ is cont on all of T and is never zero, we know

$$d(\mu) = 0.$$

But, we can relate $d(\mu)$ to $d(\gamma)$. If we compute $d(\mu)$ edge by edge we get $2\pi d(\mu) = (-\pi) + (-\pi) + 2\pi\, d(\gamma)$. To see this consider the following:

- Along the hypotenuse the winding of $u(s, s)$ is the same a the winding of $V(\gamma(s))$ is going all the way around.
- Moving along the top leg from $(L, L)$ to $(0, L)$ the $u(s, L)$ vector goes from $(1, 0)$ to $(-1, 0)$, never pointing "south". This contributes a $-\pi$ (clockwise).

- Likewise move down the vertical leg we ~~again~~ got from $(-1,0)$ to $(1,0)$ going cw and get ~~at~~ another $-\pi$ contribution.

But $$0 = -2\pi + 2\pi\, d(\gamma) \Rightarrow d(\gamma) = 1$$

as claimed.