Sections 28 and 31 (Henle)

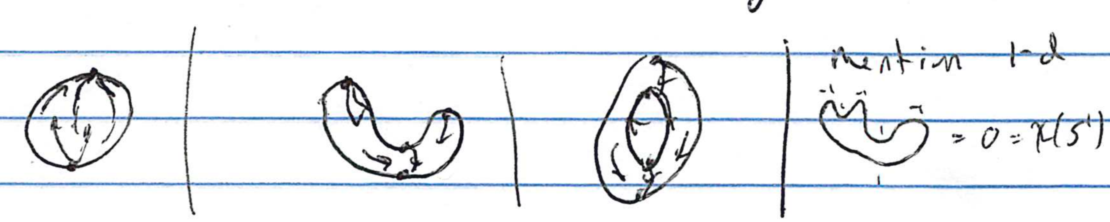
The Poincaré Index Thm*

In these section assume all surfaces are smoothly (have tangent planes) embedded in $\mathbb{R}^3$, are compact, orientable and without bdy unless stated otherwise.

<u>Def</u> A height functions is a cont function from S to $\mathbb{R}$. An easy example is $h(x,y,z)=z$, which maps S to the z-axis. It can be thought of as the (signed) distance of each pt is S from the xy-plane.

For each point $p=(x,y,z)\in S$, $\exists$ a tangent plane. Let $V(p)=$ the projection of $\nabla h_z\,(=\langle 0,0,1\rangle)$ onto the tangent plane at p. This gives a flow where flow lines move in the direction of maximum assent. (If we reverse the flow direction we get the "Chocolate syrip flow") This is called a gradient field.

<u>Examples</u>



* We will be "informal" in that our definitions and proofs will be incomplete and we stress intuition. Details can be found in books on diff top

We shall assume S is embedded such that the critical points ($V(p)=0$) are isolated. Hence, since S is compact, there are only finitely many cr pts.

Rmk ~~Examples~~ Patches that look like $f(x,y)=(x-y)^2$ or x^2y^2 are not allowed. This is not too serious a restriction since ~~as~~ a very small perturbation any such regions "go away"! Formally the set of embeddings with~~out~~ nonisolated cr. pts has measure zero.

At ~~an~~ each isolated cr. pt. P we can project the flow of an open nbhd into the tangent plane and use the index the~~i~~re as the ~~id~~ index of P. That this can be done rigorously is a result from diff. top.

Thm Let V be a grad. field on a surface S, whose cr. pt are $P_1, \dots P_n$. Then

$$I(P_1)+\dots+I(P_n)=\chi(S).$$

First we do a few more examples.

compute examples, monkey saddle

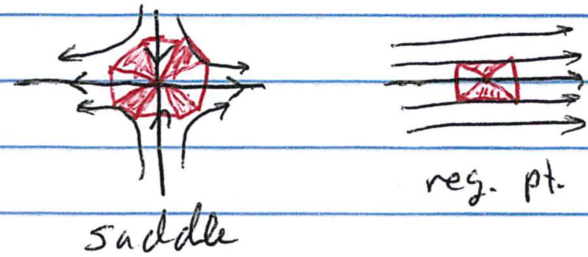
Outline of Pf Recall that for an iso'd cr. pt. $I(p) = 1 + \frac{e-h}{2}$.
But for a gradient flow $e=0$. Draw picture.
Thus we have $I(p) = 1 - \frac{h}{2}$. Note: for a reg. pt.
we can think of $h=2$ and thus $I=0$.

Let $\mathcal{T}$ be a triangulation of S, we require that $p_1 \ldots p_n$ be vertices. We assume that for each triangle the heights of the 3 vertices are distinct. The vertex with height in between the other two is called the middle vertex of that triangle.

We will also need to assume the we have enough triangles for each step below to be valid.

Now if p is a local max or min it cannot be the middle vertex of any triangle.

Suppose p has h hyperbolic sectors. We assume there are enough triangles cycling around p so that each sector contains in its interior the vertices of at least one triangle. Then the number of triangles with p as the middle vertex is equal to h.



In all cases $I(P) = 1 - \frac{1}{2}\left(\begin{smallmatrix}\text{\# of triangles for} \\ \text{which } P \text{ is the middle pt.}\end{smallmatrix}\right)$

This even works for regular pts. Thus,

$$\sum_{i=1}^{n} I(P_i) = \sum_{i=1}^{n} 1 - \frac{1}{2}\left(\begin{smallmatrix}\text{\# } \Delta\text{'s with} \\ \text{middle pt } P_i\end{smallmatrix}\right) = \star$$

Now, let $V_1 \dots V_x$ be all the vertices.
Because for none cri pts we have

$$1 - \frac{1}{2}\left(\begin{smallmatrix}\text{\#}\Delta \\ \text{mid pt } V_i\end{smallmatrix}\right) = 0 \quad \text{for } V_i \neq \text{any } P_i$$

we get

$$\star = \sum_{i=1}^{x} 1 - \frac{1}{2}\left(\begin{smallmatrix}\text{\# } \Delta\text{'s with} \\ \text{mid pt } V_i\end{smallmatrix}\right)$$

Now $\sum_{i=1}^{x} (\text{\# } \Delta\text{'s with mid pt } V_i) = \text{\# } \Delta\text{'s} = F,$

because each Δ one middle vertex. Thus

$$\star = V - \frac{1}{2}F.$$

Finally, for any triangulation each Δ has 3 edges and each edge is part of 2 triangles. Thus,

$$\frac{F}{E} = \frac{2}{3} \quad \text{or} \quad 3F = 2E.$$

Hence, $F = 2E - 2F$ and

$$\star = V - \frac{1}{2}(2E - 2F) = V - E + F = \chi(S).$$

$\blacksquare$

For the next result we need a generalized version of the index lemma (~~Thm~~ page 47 as applied to isolated cr. pt.s in section 9.)

<u>Index Lemma</u> Let S be an orientable compact surface with bdy. Let $\mathcal{K}$ be an oriented triangulation of S whose vertices are labeled ~~with~~ A, B or C. The <u>content</u> C is the number of ABC triangles counted by orientation: +1 if ABC agrees with the orientation, −1 if not. The <u>index</u> I is the sum of AB/BA edges in the bdy counted + if AB agrees with the orientation, −1 if not. Then $C = I$.

The proof goes the same as was done in section 7.

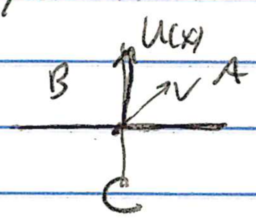
Section 31 extends this result to any cont. v.f. on a surface S, with no cr. pts

Thm Let V be a cont. v.f. on a surface S with isolated cr. pts, $P_1 \dots P_n$. Then

$$\sum_{i=1}^{n} I(P_i) = \chi(S).$$

Outline of Pf Choose a grad field U on S s.t. its cr pts, $Q_1 \dots Q_m$ are distinct from $P_1 \dots P_n$.

For each pt x of $S - \{P_1 \dots P_n, Q_1 \dots Q_m\}$ we give a label of A, B or C as follows. Use $U(x)$ as the "y axis" and proceed as in Chapter 2: labeling x A, B or C depending on the direction of $V(x)$ wrt $U(x)$.



Remove the interiors of "small" disjoint closed disks containing $P_1 \dots P_n$, $Q_1 \dots Q_m$. Call this surface with bdy S'. Since $V(x)$ is never zero, for a fine enough triangulation there will be no ABC triangles. Thus the content, C, is zero, using the labels defined above for the vertices.

The index of this labeling can be broken down by bdy component. Let $I^*(x)$ be the index of the bdy component associated to x for each $x \in \{P_1 \ldots P_n, Q_1 \ldots Q_m\}$.

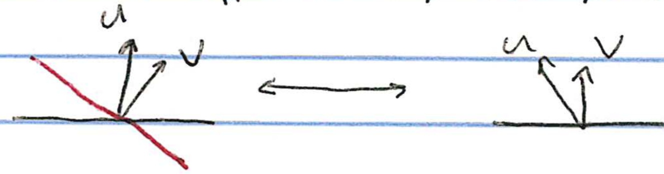
~~A~~ If the ~~number of~~ vertices along a bdy comp associated to P_i are close enough together then since for a small enough disk the vectors of U are "nearly parallel" we have $I^*(P_i) = I(P)$, $i = 1 \ldots n$.

Discuss "nearly parallel".

However this need not be true for the Q_i's. The trick is we will relabel the vertices of each bdy comp assoc. to a Q_i using the V vectors as the y axis and letting the ~~U vector~~ ~~slot~~ quadrant of the U vector determine the label. Since the vectors of V are "nearly parallel" we can compute $I(Q_i)$ in this way. But these new labels may create ABC Δ's and has no ~~obv~~ obvious relation to $I^*(Q_i)$. What to do?

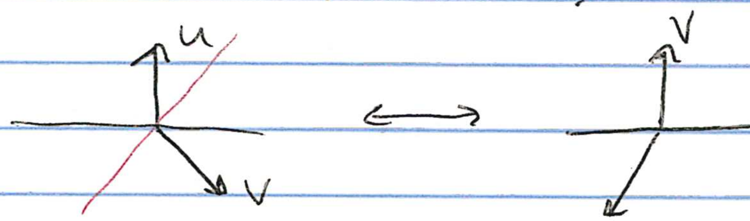
We study the two labeling schemes.

First suppose $\measuredangle \leq 90°$.



A becomes B and B becomes A.

What if $\measuredangle > 90°$?



C remains C.

Therefore $I^*(Q_i) = -I(Q_i)$.

We put this together:

$$I^*(P_1) + \cdots + I^*(P_n) + I^*(Q_1) + \cdots + I^*(Q_m) = I = C = 0$$

$$\Rightarrow \quad I(P_1) + \cdots + I(P_n) - I(Q_1) - \cdots - I(Q_m) = 0$$

$$\Rightarrow \quad I(P_1) + \cdots + I(P_n) = \underbrace{I(Q_1) + \cdots + I(Q_m)}_{\chi(S)} =$$

by our thm about gradient flows!

$\boxed{/\!/}$