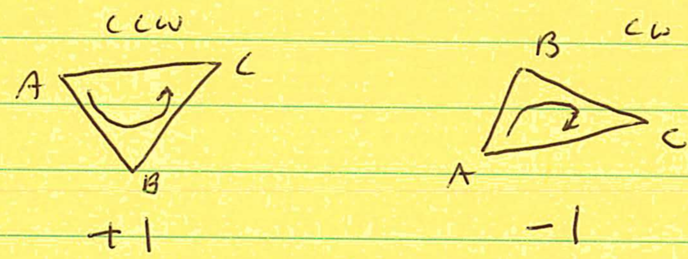
Index Lemma

Let P be a polygon in $\mathbb{R}^2$. Triangulate P and label the vertices with A, B or C, any way you like. We going to define two quantities associated to such a labeled triangulation, the content, C, and the index, I. Then we will prove they are equal.

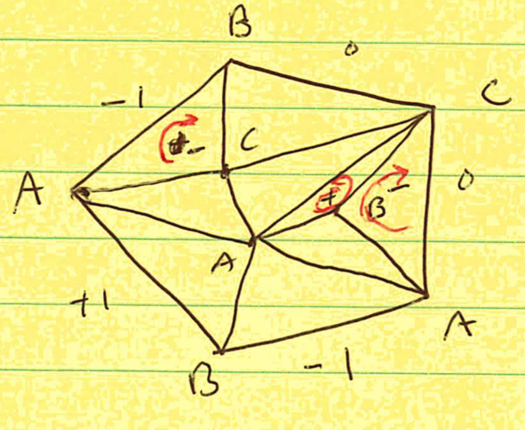
Def Assign an orientation to each complete (ABC) triangle in the triangulation as follows:



The content, C, of the triangulation is the sum of these orientations.

Def Now look only at the exterior edges of the triangulation of P. Going around ccw label AB edges $+1$, BA edge -1 and any others 0. The index, I, is the sum of these labels.

Ex

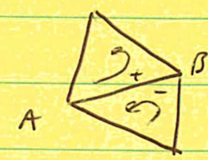


$C = -1 + 1 - 1 = -1$

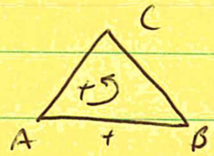
$I = -1 + 1 - 1 = -1.$

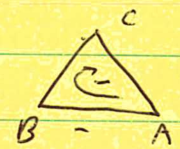
Index Lemma $C = I$. (Should remind you of Green's Thm)

Pf For each triangle go around ccw and count AB edges as $+1$ and BA edges as -1. All others are zero. Add these up. Call this number S. Notice that $S = I$ because all interior AB/BA edges get counted twice, once with $+1$, once with -1.

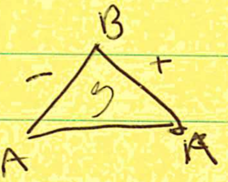


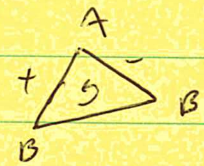
I claim $S = C$ as well. A positive oriented triangle gives a $+1$ ~~to~~ C and to S. A neg. oriented triangle gives a -1 to C and to S.





All other triangles give zero to C, by definition, but also give zero to S. This is obvious if there are no AB/BA edges. For AAB and ABB triangles the net result is zero:





Therefore, $S = C$. Thus $I = C$. $\square$