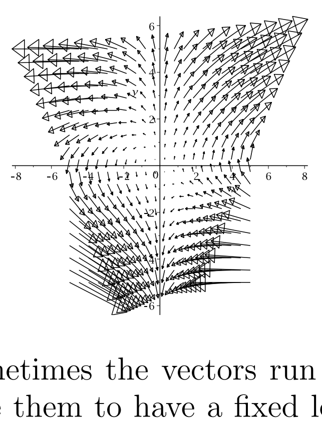
MATH 405
Systems of First Order Ordinary Differential Equations
Course Overview

In this course we will study systems of first order ordinary differential equations. These take the form

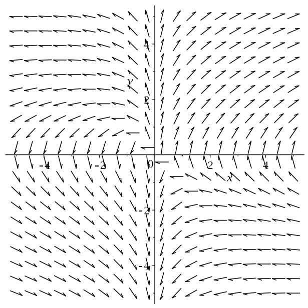
$$
\begin{aligned}
x_1'(t) &= F_1(x_1, x_2, \ldots, x_n, t), \\
x_2'(t) &= F_2(x_1, x_2, \ldots, x_n, t), \\
&\vdots \\
x_n'(t) &= F_n(x_1, x_2, \ldots, x_n, t).
\end{aligned}
$$

If there is no explicit dependence on time, t, in the F_i's, then the system is said to be **autonomous**. In this case we get a **vector field**.

Example. Let $x'(t) = x(t)y(t)$ and $y'(t) = x(t) + y(t)$. The vector field is below.



Sometimes the vectors run over each other and it is useful to normalize them to have a fixed length. This is called a **direction field.** See below for the direction field for the same example.



But, with the direction field is it not clear the the vectors are getting smaller near to the x-axis. (If you leave off the arrow heads, it is called a slope field.)

Chapter 7. Here we focus on the special case

$$\begin{aligned} x_1'(t) &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + g_1(t), \\ x_2'(t) &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + g_2(t), \\ &\vdots \\ x_n'(t) &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + g_n(t). \end{aligned}$$

In fact the $g_i(t)$ terms are only covered in the last section, 7.9. When the g_i terms are absent the autonomous systems of this form is said to be **homogeneous**. If we let $A = [a_{ij}]$, $\mathbf{x} = [x_1 x_2 \cdots x_n]^T$ and $\mathbf{g} = [g_1 g_2 \cdots g_n]^T$ the system above is equivalent to

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}.$$

Solutions are found by working with the **eigenvalues** and **eigenvectors** of the matrix. These are defined in Section 7.3. Most of the time we will be studying 2x2 homogeneous systems.

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

In this case the solution methods form a trichotomy. There will be two eigenvalues.

- If the eigenvalues are real and distinct, a certain method is used.
- If the eigenvalues are real but repeated, the method may be modified.
- If they eigenvalue are complex conjugates, the method is different.

This should remind you of the situation you encountered if solving second order equations of the form $ay'' + by' + cy = 0$ back in Chapter 3.

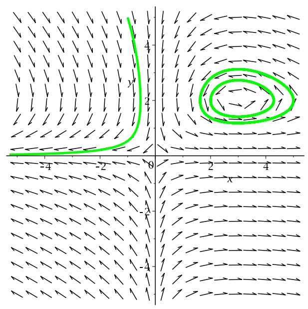
Chapter 9. Now we consider some nonlinear systems. We stick to autonomous ones.

The basic method is to first find the **critical points** where are the x_i' are zero. Then one finds an approximate linear system at each critical point and build up an understanding of the system from these pieces. We also study **periodic solutions**. These are solutions curves that form closed loops. At the end of Chapter 9 we explore the 3x3 Lorenz Equations and the beginning of Chaos Theory.

Example. Below we show the direction field for the system,

$$x' = x - xy/2,$$
$$y' = -3y/4 + xy/4,$$

together with some solution curves.



Here is a YouTube video of the famous/infamous Lorenz system.

https://www.youtube.com/watch?v=FYE4JKAXSfY

Study hard and have a great semester!