

7.1 Intro. (Converting systems to single higher order eq's)

Problem

$$\text{Given } x'(t) = a x(t) + b y(t)$$

$$y'(t) = c x(t) + d y(t)$$

convert to 2nd order linear in one dependent variable.

Solution

Solve second equation for x and substitute into the first equation.

$$x = \frac{y' - dy}{c} \quad (\text{assuming } c \neq 0)$$

$$\left(\frac{y' - dy}{c} \right)' = a \left(\frac{y' - dy}{c} \right) + by$$

$$\frac{y'' - dy'}{c} = \frac{a}{c} (y' - dy) + by$$

$$y'' - dy' = ay' - ady + bcy$$

$$y'' - (a+d)y' + (ad-bc)y = 0.$$

$$\text{Let } M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \quad \det M = ad - bc \quad \text{tr } M = a + d$$

$$y' - (\text{tr } M)y + (\det M)y = 0.$$

Note: If $c = 0$, $y = C_1 e^{dt}$, then we have a one degree equation in one dependent variable.

$$x' = ax + C_1 e^{dt}. \quad (C = b C_1).$$

Example 1 Solve $x' = 2x - 3y$ $x(0) = 1$
 $y' = x - 2y$ $y(0) = 2$

Solution $M = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$ $\text{tr } M = 0$ $\det M = -4 - (-3) = -1.$

Thus, we have $y'' - 0y' - y = 0$

$$r^2 - 1 = 0$$

$$r = \pm 1$$

$$y = C_1 e^t + C_2 e^{-t}.$$

$$x = y' + 2y = (C_1 e^t + C_2 e^{-t})' + 2(C_1 e^t + C_2 e^{-t})$$

$$= 3C_1 e^t + C_2 e^{-t}$$

$$x(0) = 1 \Rightarrow 3C_1 + C_2 = 1$$

$$y(0) = 2 \Rightarrow \underline{C_1 + C_2 = 2}$$

$$2C_1 = -1 \Rightarrow C_1 = -\frac{1}{2}$$

$$C_2 = 2 - C_1 = \frac{5}{2}$$

$$x = -\frac{3}{2}e^t + \frac{5}{2}e^{-t}$$

$$y = -\frac{1}{2}e^t + \frac{5}{2}e^{-t}$$

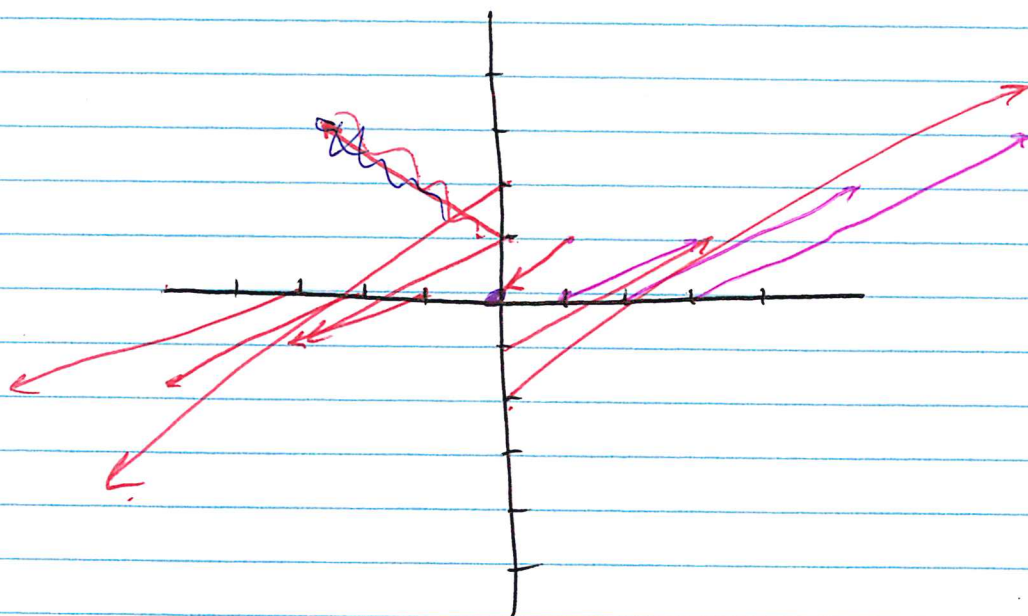
Next we look at the vector field for this system.

$$x' = 2x - 3y$$

$$y' = x - 2y$$

(x, y)	$\langle x', y' \rangle$
$(0, 0)$	$\langle 0, 0 \rangle$
$(1, 0)$	$\langle 2, 1 \rangle$
$(2, 0)$	$\langle 4, 2 \rangle$
$(3, 0)$	$\langle 6, 3 \rangle$
$(-1, 0)$	$\langle -2, -1 \rangle$
$(0, 1)$	$\langle -3, -2 \rangle$
$(0, 2)$	$\langle -6, -4 \rangle$
$(1, 1)$	$\langle -1, -1 \rangle$
$(1, 2)$	$\langle -4, -3 \rangle$

equilibrium point
(also called:
critical pt
rest pt
fixed pt)



Let's try it on a computer!

I. Using <https://www.geogebra.org/m/QPE4PaDZ>.

II. Using Maple

Vector Field $\langle 2x - 3y, x - 2y \rangle$

$V_x(x,y) = 2x - 3y$

$V_y(x,y) = x - 2y$

$x_{\min} = -5$

$x_{\max} = 5$

$y_{\min} = -5$

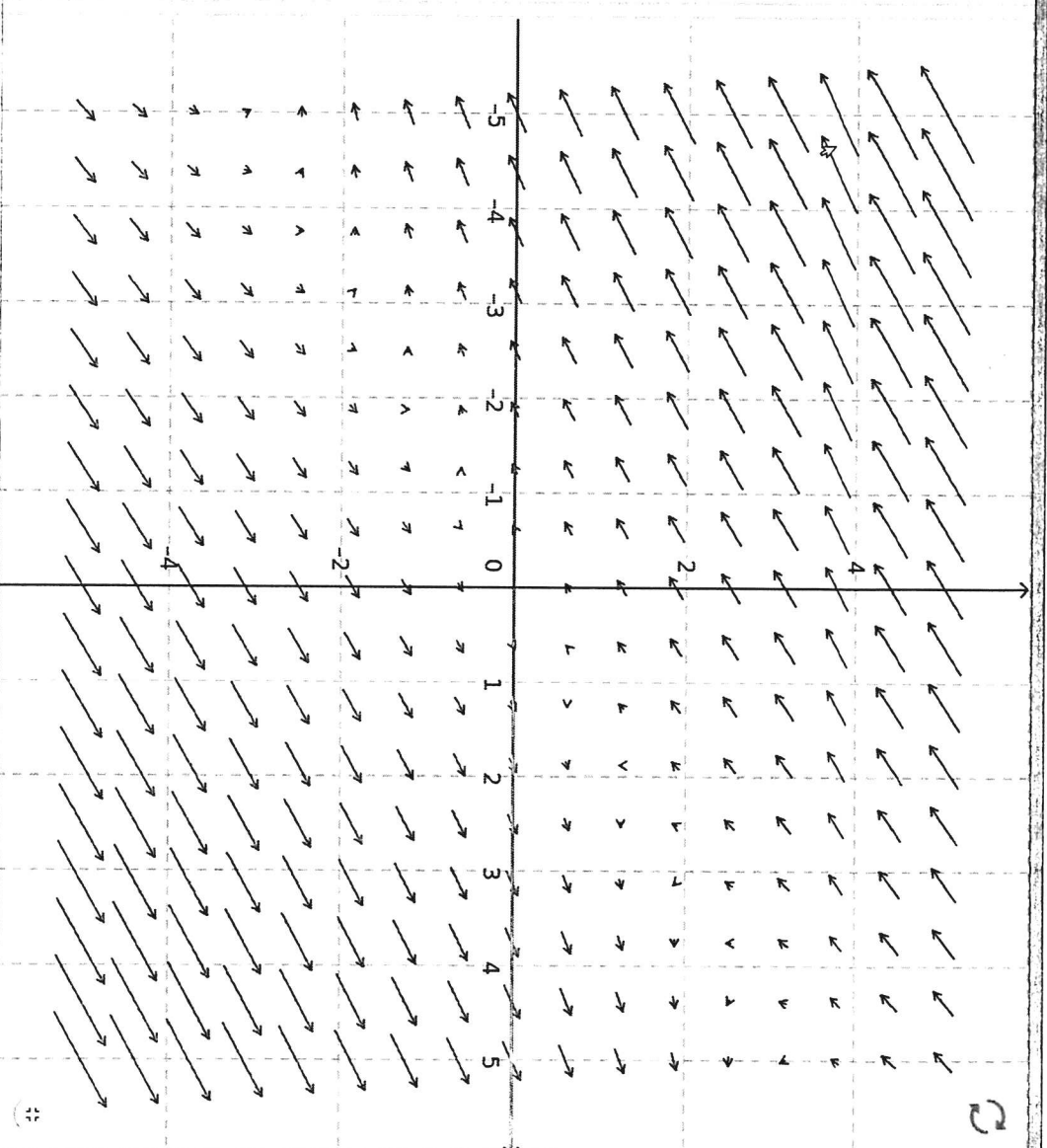
$y_{\max} = 5$

$x_n = 8$

$y_n = 8$

$v = 0.02$

$v_h = 0.09$



Geogebra

www.geogebra.org/m/QPE4PaDZ

Example 2 Solve $x' = 3x - y$ $x(0) = 1$
 $y' = x + 2y$ $y(0) = 0$

Solution

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

"
 $\det M = 7$
 $\text{tr } M = 5$

Thus, we have

$$y'' - 5y' + 7y = 0$$

$$r^2 - 5r + 7 = 0$$

$$r = \frac{5 \pm \sqrt{25 - 4 \cdot 1 \cdot 7}}{2} = \frac{5 \pm i\sqrt{3}}{2}$$

Thus,

$$y(t) = C_1 e^{\frac{5}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right) + C_2 e^{\frac{5}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right)$$

$$x(t) = y' - 2y =$$

$$\begin{aligned} & \frac{5}{2} C_1 e^{\frac{5}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{2} C_1 e^{\frac{5}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{5}{2} C_2 e^{\frac{5}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{\sqrt{3}}{2} C_2 e^{\frac{5}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right) \\ & - 2 C_1 e^{\frac{5}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right) - 2 C_2 e^{\frac{5}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) \end{aligned}$$

$$= \left[\frac{1}{2} C_1 - \frac{\sqrt{3}}{2} C_2 \right] e^{\frac{5}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right) + \left[\frac{\sqrt{3}}{2} C_1 + \frac{1}{2} C_2 \right] e^{\frac{5}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right)$$

$$y(0) = 0 \Rightarrow C_2 = 0.$$

$$x(0) = 1 \Rightarrow x(0) = \frac{\sqrt{3}}{2} C_1 = 1 \Rightarrow C_1 = \frac{2}{\sqrt{3}}.$$

Thus, $x = \frac{1}{\sqrt{3}} e^{\frac{5}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right) + e^{\frac{5}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right)$

$$y = \frac{2}{\sqrt{3}} e^{\frac{5}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right).$$

Vector Field $< 3x - y, x + 2y >$

$V_x(x,y) = 3x - y$

$V_y(x,y) = x + 2y$

$x_{\min} = -5$

$x_{\max} = 5$

$y_{\min} = -5$

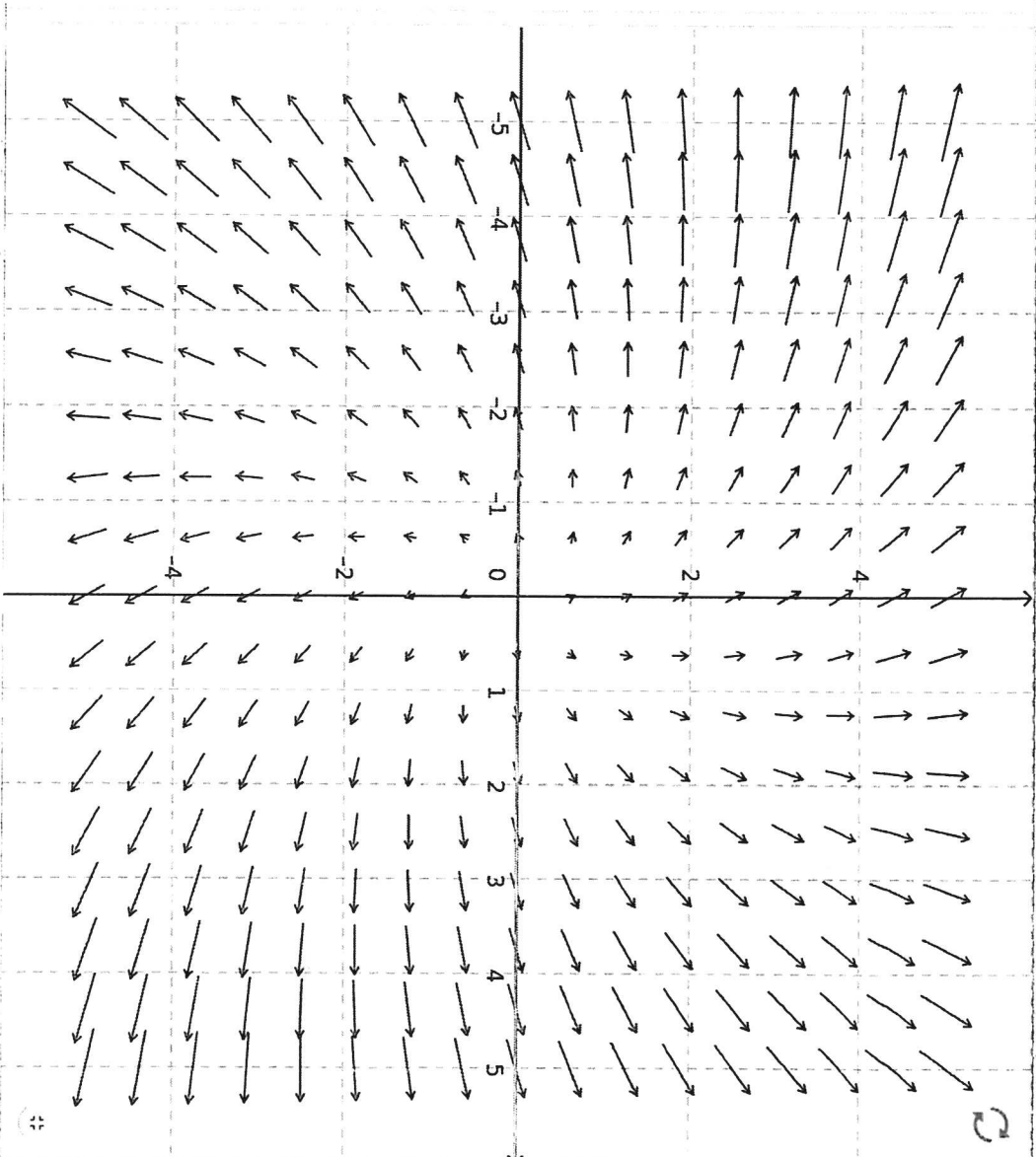
$y_{\max} = 5$

$x_n = 8$

$y_n = 8$

$v = 0.02$

$v_h = 0.09$



Gesamte
Vector Field für Example 2.

If the system is nonhomogeneous we can still convert to a single second degree equation:

$$\begin{aligned}\text{Suppose } x' &= ax + by + f(t) \\ y' &= cx + dy + g(t)\end{aligned}$$

$$\text{Then } x = \frac{y' - dy - g(t)}{c}$$

$$\text{and } x' = \frac{y'' - dy' - g'(t)}{c}$$

$$\text{Thus, } \left[\frac{y'' - dy' - g'(t)}{c} \right] = a \left[\frac{y' - dy - g(t)}{c} \right] + by + f(t)$$

$$y'' - dy' - g' = ay' - ady - ag + by + cf$$

$$y'' - (a+d)y' + (ad-bc)y = g' - ag + cf$$

Example 3 $x' = -x + 2y + \cos t$ Find general solution.
 $y' = 2x - y + \sin t$

Solution $\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ trace $= -2$ det $= -3$.

We have $y'' + 2y' - 3y = \cos t + \sin t + 2 \cos t$
 $= 3 \cos t + \sin t.$

homogeneous part: $y_h'' + 2y_h' - 3y_h = 0$

$$r^2 + 2r - 3 = 0$$

$$(r+3)(r-1) = 0 \quad r = -3, 1$$

$$y_h = C_1 e^{-3t} + C_2 e^t$$

particular solution

Let $y_p = A \sin t + B \cos t.$

$$\Rightarrow y_p' = A \cos t - B \sin t.$$

$$\Rightarrow y_p'' = -A \sin t - B \cos t.$$

Next,

$$y_p'' + 2y_p' - 3y_p = \underbrace{(-A - 2B - 3A)}_{=1} \sin t + \underbrace{(-B + 2A - 3B)}_{=3} \cos t$$

$$4A + 2B = 1$$

$$2A - 4B = 3$$

$$10A = 1$$

$$A = \frac{1}{10}$$

$$B = -\frac{7}{10}$$

General solution for y is,

$$y(t) = C_1 e^{-3t} + C_2 e^t + \frac{1}{10} \sin t - \frac{7}{10} \cos t.$$

General solution for x is

$$\begin{aligned} x(t) &= \frac{y' + y - \sin t}{2} = \frac{-2C_1 e^{-3t} + 2C_2 e^t + (\frac{1}{10} \cos t + \frac{7}{10} \sin t) + (\frac{1}{10} \sin t - \frac{7}{10} \cos t) - \sin t}{2} \\ &= -2C_1 e^{-3t} + 2C_2 e^t - \frac{3}{10} \cos t - \frac{1}{10} \sin t \end{aligned}$$

We can also go the other way. Given

$$ay'' + by' + cy = 0$$

it is easy to convert to a 2×2 linear system:

$$\text{Let } x(t) = y'(t).$$

$$\text{Then } x' = y'' = -\frac{b}{a}y' - \frac{c}{a}y.$$

Thus, we have

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -\frac{b}{a} & -\frac{c}{a} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$