Section 7.2 Matrices and Linear Algebra

An $m \times n$ matrix is a rectangular array of numbers, real or complex, or functions, with m rows and n columns.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4i \end{bmatrix} \quad B = \begin{bmatrix} 7t & t+2 \\ \sin t & 0 \\ t^2 & 6t \end{bmatrix} \quad C = \begin{bmatrix} 1 \\ 2 \\ t \end{bmatrix} \quad R = \begin{bmatrix} 6 & 1 & 9 \end{bmatrix}$$

2×2 $\quad$ 3×2 $\quad$ 3×1 column vector $\quad$ 1×3 row vector.

We may write $A = [a_{ij}]$, $i = 1 \ldots m$, $j = 1 \ldots n$.

Here are three common matrix operations:

Transpose: $[a_{ij}]^T = [a_{ji}]$ Turns an $m \times n$ matrix into an $n \times m$ matrix.

$$\begin{bmatrix} 1 & -2 & 3i \\ 4 & 5i & 6t \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ -2 & 5 \\ 3i & 6t \end{bmatrix}$$

Conjugation: For complex numbers $\overline{a+bi} = a - bi$.
For a matrix $\overline{[a_{ij}]} = [\overline{a_{ij}}]$.

Adjoint:

$$A^* = \overline{A}^T,$$

Note: If $A^* = A$, then A is self-adjoint or Hermitian. These have special properties.

Algebraic Operations

Addition: If A and B have the same size/dimensions and $A = [a_{ij}]$, $B = [b_{ij}]$, then

$$A + B = [a_{ij} + b_{ij}].$$

Multiplication of matrices is based on the vector dot product. Let A be $m \times n$ and B be $p \times q$. If $n=p$ then AB is formed by taking the dot products of the rows of A with the columns of B.

Examples:

$$\begin{bmatrix} 2 & -1 \\ i & 3 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ i & 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 4+(-1)1 & 2\cdot 0+(-1)1 \\ i\cdot 4+3\cdot 1 & i\cdot 0+3\cdot 1 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 3+4i & 3 \end{bmatrix}.$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 4 \\ -1 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x+2y+z \\ 2x+3y+4z \\ -x+2y+2y \end{bmatrix}.$$

In general matrix multiplication is not commutative. Indeed if AB is defined (i.e. $n=p$), BA may not be defined since m might not equal p. Even square matrices of the same size need not commute:

$$\begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ i & 3 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ 2+i & 2 \end{bmatrix} \neq \begin{bmatrix} 7 & -1 \\ 3+4i & 3 \end{bmatrix}.$$

Identity Matrix I is the square matrix given by

$$a_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}, \quad i,j=1\ldots n.$$

For $n=3$, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. You can check for any square matrix A

$$AI = IA = A.$$

Zero Matrix 0 = matrix of all zeros.

Powers For square matrices we can define powers: Let A be $n \times n$. Then

$$A^2 = AA, \quad A^3 = AAA, \text{ etc.} \quad A^1 = A \text{ and } A^0 = I.$$

Examples
$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$$

Scalar Multiplication Let $a \in \mathbb{R}$ or $\mathbb{C}$. Let $A = [a_{ij}]$

Then $aA = [a \cdot a_{ij}]$.

Exponential Function of a Matrix. Let A be $n \times n$.

$$\exp(A) = e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

It can be shown that this limit exists.

Algebraic Properties

$A + 0 = 0 + A = A$. $A + (-A) = 0$. $A + B = B + A$.
$a(A+B) = aA + aB$.

$A(BC) = (AB)C$ (associative)
$A(B+C) = AB + AC$
$(A+B)C = AC + BC$
$a(AB) = (aA)B = A(aB)$.

$AI = IA = A$.

Determinants

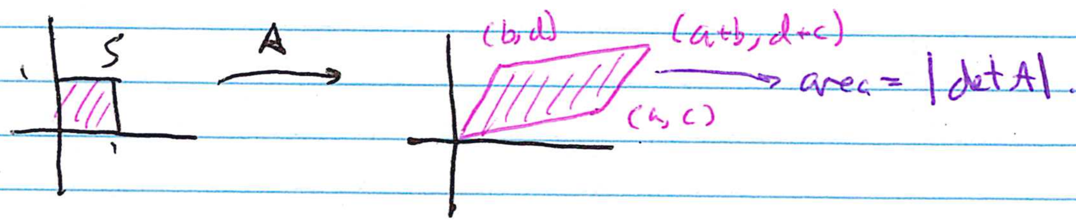
An $n \times n$ real matrix A can be thought of as a function from $\mathbb{R}^n$ to $\mathbb{R}^n$, $y = Ax$, where x and y are $n \times 1$ column vectors.

Let $S = [0,1]^n = \{(x_1, x_2, \ldots x_n) \in \mathbb{R}^n \mid 0 \le x_i \le 1\}$. It is a unit cube in $\mathbb{R}^n$. The image $A(S)$ is some parallelopiped in $\mathbb{R}^n$. Its "signed volume" is $\det A$.

There are several ways to compute $\det A$ (also written $|A|$)

(Determinants are still useful when A has complex entries, but the geometric idea is less clear.)

__Example__ For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.



__Example__ For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ we have

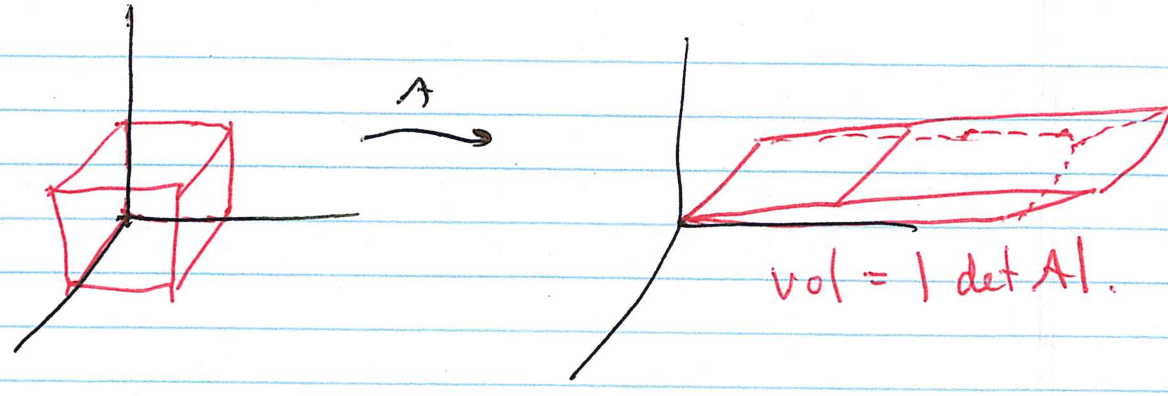
$$\det A = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(expasion along 1st row)

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

(expasion along 1st column)

You can expand along any row or column to get $\det A$.



In general, Let A be $n \times n$. Let A_{ij} = the det of the $(n-1) \times (n-1)$ submatrix obtained from A by deleting the ith row and the jth column. Then

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} A_{ij} \quad \text{for any } i = 1, \dots n.$$

(expansion along i^{th} row)

$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} A_{ij} \quad \text{for any } j = 1 \dots n$$

(expansion along j^{th} row)

Tricks
- det A is unchanged if a multiple of one row or column is added to another row or column, resp.
- det A changes by a factor of x if a row or column is multiplied by x.
- det changes sign if two rows or two columns are switched.

Example 1 (Upper triangular matrices). In the following example (from Len Evans', *A Brief Course in Linear Algebra*) identify the operation being used.

$$\begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 3 & 0 & 1 & 1 \\ -1 & 6 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & -6 & 4 & -2 \\ 0 & 8 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 7 & 4 \\ 0 & 0 & -5 & -5 \end{vmatrix} =$$

$$-5\begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 7 & 4 \\ 0 & 0 & 1 & 1 \end{vmatrix} = +5\begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 7 & 4 \end{vmatrix} = +5\begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{vmatrix}.$$

The last matrix is an upper triangular matrix. Its determinant is especially easy to compute.

$$\begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{vmatrix} = 1\cdot\begin{vmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{vmatrix} = 1\cdot 2\cdot\begin{vmatrix} 1 & 1 \\ 0 & -3 \end{vmatrix} = 1\cdot 2\cdot 1\cdot(-3) = -6.$$

Thus the determinant of the original matrix is $5\cdot(-6) = -30$.

Facts $\det AB = \det A \cdot \det B$.

$\det A = \det A^T$.

Inverses Let A be $n \times n$. If B is $n \times n$ and $AB = I$, then it can be shown that $BA = I$. We call B the *inverse* of A and write $B = A^{-1}$. Inverses are unique. A is said to be *invertible* or *nonsingular* if A^{-1} exists.

Fact: A has an inverse if and only if $\det A \neq 0$.
In this case $\det A^{-1} = \dfrac{1}{\det A}$.

Fact Let A be $n \times n$ and b $n \times 1$ with x $n \times 1$ unknown.
Then

$$Ax = b$$

has a unique solution if and only if $\det A \neq 0$.
This solution is $x = A^{-1} b$.

There is a formula for A^{-1}, but it is hard to use:

$$\text{Let } A^{-1} = B = [b_{ij}]. \text{ Then } b_{ij} = \frac{(-1)^{i+j} A_{ji}}{\det A}.$$

There is an easier way to find A^{-1} using row operations. See Example 2 in Section 7.2.

Complex Vectors

When working with vectors with complex entries the following inner product is useful:

Let $x = (x_1, \dots x_n)$, $y = (y_1, \dots, y_n)$ be complex vectors.

There inner product is $(x, y) = \sum_{i=1}^{n} x_i \bar{y}_i = x \cdot \bar{y}$.

If they are column vectors $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

we write $(x, y) = x^T \bar{y}$.

Properties

$$(x, y) = \overline{(y, x)}$$
$$(x, y+z) = (x, y) + (x, z)$$
$$(x+y, z) = (x, z) + (y, z)$$
$$(ax, y) = a(x, y)$$
$$(x, ay) = \bar{a}(x, y)$$

Define $\|x\|^2 = (x, x) = x^T \bar{x} = \sum_{i=1}^{n} |x_i|^2$.

(Recall for complex numbers $|a+bi| = \sqrt{a^2+b^2}$

$= [(a+bi)(a-bi)]^{1/2}$.

Calculus and Matrices

Let $A = [a_{ij}(t)]$ be a matrix of functions.

$$\frac{dA}{dt} = \left[\frac{d\,a_{ij}(t)}{dt} \right] \qquad \int A\,dt = \left[\int a_{ij}(t)\,dt \right]$$

Let $B = [b_{ij}(t)]$ and $C = [c_{ij}]$ (constants).

Then $\dfrac{dC}{dt} = 0 \qquad \dfrac{dCA}{dt} = C\dfrac{dA}{dt}$

$$\frac{d(A+B)}{dt} = \frac{dA}{dt} + \frac{dB}{dt}.$$

$$\frac{dAB}{dt} = A\frac{dB}{dt} + \frac{dA}{dt}B.$$