Section 7.3

$n \times n$ systems of linear equations:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}$$

or, $Ax = b$.

If b is the zero vector, we say $Ax = 0$, is homogeneous. If $b \neq 0$ (has at least one nonzero entry), then we say $Ax = b$ is nonhomogeneous.

If $\det A \neq 0$, then A^{-1} exists, and $Ax = b$ has a unique solution, $x = A^{-1}b$.

If $\det A = 0$, then

$Ax = 0$ has infinitely many solutions

and $Ax = b \neq 0$ has ~~no~~ either no solutions or infinitely many.

Vector Spaces

Let $V \subset \mathbb{R}^n$ or $\mathbb{C}^n$. If the two closure axioms hold then V is called a vector space.

① If $v, w \in V$, then $v+w \in V$.

② If $\alpha \in \mathbb{R}$ (or $\mathbb{C}$), and $v \in V$ then $\alpha v \in V$.

In $\mathbb{R}^2$, lines going through the origin are vector spaces. So are $\mathbb{R}^2$ itself and $\{0\}$.

In $\mathbb{R}^3$, lines and planes going through the origin are vector spaces. So are $\mathbb{R}^3$ itself and $\{0\}$.

If F is a set of functions going from $\mathbb{R}$ to $\mathbb{R}$, then F is said to be a vector space if the two closure axioms hold true.

Example The solution set to $y'' + y = 0$ is the vector space $\{C_1 \cos x + C_2 \sin x \mid C_1, C_2 \in \mathbb{R}\}$.

Linear Dependence and Independence

A set of vectors $\{v_1, \dots, v_k\}$ is said to be linearly independent

if

$$c_1 v_1 + c_2 v_2 + \cdots c_k v_k = 0 \qquad (*)$$

has only one solution: $c_1 = c_2 = \cdots = c_k = 0$.

If there is a nontrivial solution $\{v_1 \cdots v_k\}$ is lineally dependent.

We can write $(*)$ as a matrix ~~format~~ equation with $A = [v_1\ v_2\ \ v_k]$, with the v_i's as column vectors, and $c = [c_1\ c_2 \cdots c_k]^T$ as

$$Ac = 0.$$

If ~~the~~ A is $n \times n$ then $\{v_1 \cdots v_n\}$ is LI iff $\det A \neq 0$.

Example Solve $\begin{aligned} 2x - y &= 4 \\ x + 3y &= 7 \end{aligned}$ or show there are no solutions.

$$\left[\begin{array}{cc|c} 2 & -1 & 4 \\ 1 & 3 & 7 \end{array}\right]$$

$$\begin{array}{ccc} 0 & -7 & -10 \\ 1 & 3 & 7 \end{array}$$

$$\begin{array}{ccc} 1 & 3 & 7 \\ 0 & -7 & -10 \end{array}$$

$$\begin{array}{ccc} 1 & 3 & 7 = 49/7 \\ 0 & 1 & 10/7 \end{array}$$

pivots

$$\begin{array}{ccc} 1 & 0 & \frac{19}{7} \\ 0 & 1 & 10/7 \end{array}$$

This is reduced row echelon form

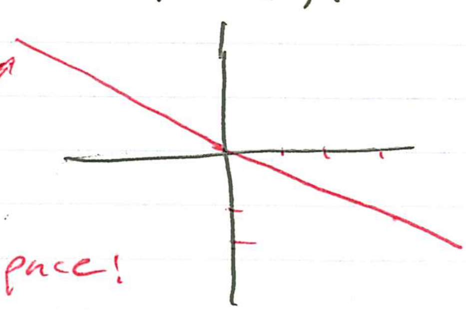
$$x = \frac{19}{7}$$

$$y = \frac{10}{7}$$

Example $\begin{aligned} 2x + 3y &= 2 \\ 4x + 6y &= 3 \end{aligned}$ no solutions

Example $\begin{aligned} 2x + 3y &= 0 \\ 4x + 6y &= 0 \end{aligned}$ ∞-many solutions.

$$y = -\frac{2}{3}x$$



Solution set is a vector space!

Example X Find all solutions to $\begin{bmatrix} 1 & -4 & -6 \\ 4 & 11 & 12 \\ -3 & -6 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Use row ops to put in to reduced row echelon form RREF

$$\begin{matrix} 1 & -4 & -6 & \\ 0 & 27 & 36 & R2 - 4 \cdot R1 \\ 0 & -18 & -24 & R3 + 3R1 \end{matrix}$$

$$\begin{matrix} 1 & -4 & -6 & \\ 0 & 3 & 4 & R2 \div 9 \\ 0 & 3 & 4 & R3 \div 6 \end{matrix}$$

$$\begin{matrix} 1 & -4 & -6 & \\ 0 & 1 & 4/3 & R2 \div 3 \\ 0 & 0 & 0 & R3 - R2 \end{matrix}$$

$$\begin{matrix} 1 & 0 & -\frac{2}{3} & R1 + 4R2 \\ 0 & 1 & \frac{4}{3} & \\ 0 & 0 & 0 & \end{matrix}$$

This is RREF.

$x = +\frac{2}{3}z$

$y = -\frac{4}{3}z$

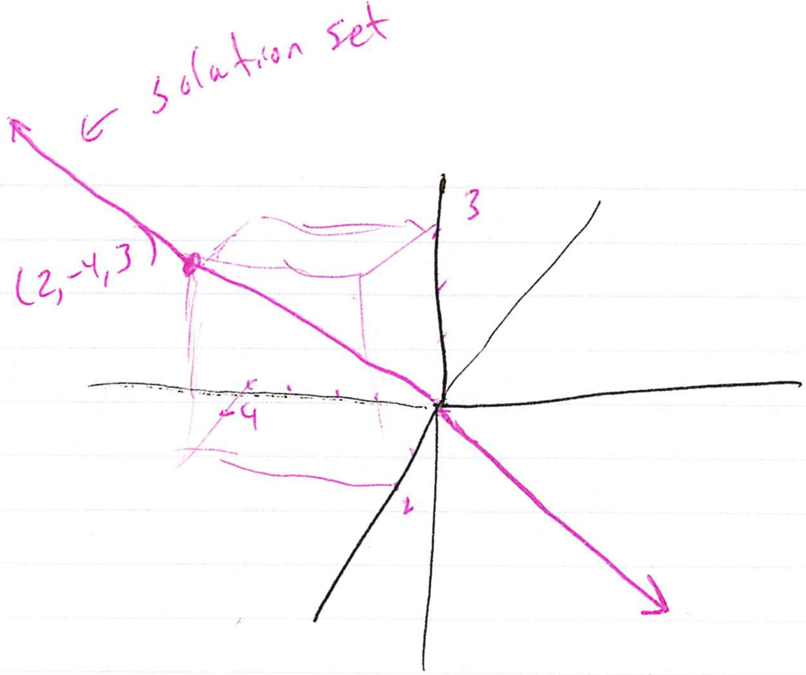
$z = z$ ← free variable

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix} z$$

Solution set is the span of $\begin{bmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$

$$= \left\{ z \begin{bmatrix} 2/3 \\ -4/3 \\ 1 \end{bmatrix} \;\middle|\; z \in \mathbb{R} \right\} = \left\{ z \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix} \;\middle|\; z \in \mathbb{R}. \right\}$$

It is a vector space. It also called the null space of the matrix.


← solution set
(2,-4,3)
3
-4
2

Example 4 Find solution set for $\begin{bmatrix} -2 & -4 & -6 \\ 4 & 8 & 12 \\ -3 & -6 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$\begin{matrix} -2 & -4 & -6 \\ 4 & 8 & 12 \\ -3 & -6 & -9 \end{matrix}$$

$$\begin{matrix} 1 & 2 & 3 & & R1 \div -2 \\ 1 & 2 & 3 & & R2 \div 4 \\ 1 & 2 & 3 & & R3 \div -3 \end{matrix}$$

$$\begin{matrix} 1 & 2 & 3 & & \\ 0 & 0 & 0 & & R2 - R1 \\ 0 & 0 & 0 & & R3 - R1 \end{matrix}$$

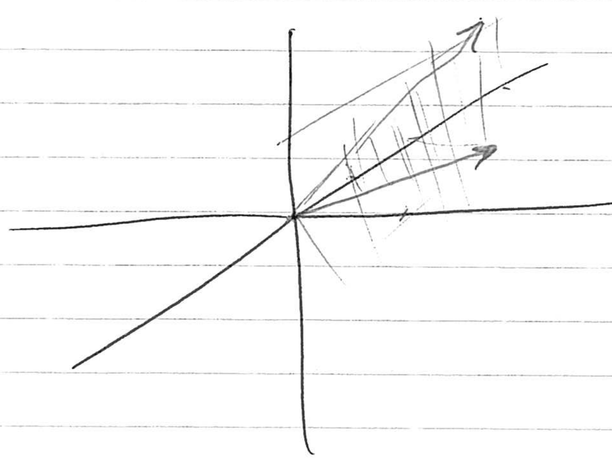
$x + 2y + 3z = 0$ It is a plane in $\mathbb{R}^3$

$$\begin{aligned} x &= -2y - 3z \\ y &= \quad\ y \\ z &= \qquad\quad z \end{aligned}$$

free variables.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} z$$

Solution set is span of $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$



Solution set is this plane in $\mathbb{R}^3$. It is a vector space.

Eigenvalues, Eigenvectors and Eigenspaces

Eigen is the German word for characterist.

Consider the problem:

$$Ax = \lambda x \qquad (*)$$

where A is $n \times n$, is given, x is $1 \times n$ column vector of unknows

and λ is an unknown real of complex number.

Geometrically, A maps the vector space $\{rx \mid r \in \mathbb{R}\}$ (if λ is real, $\neq 0$) onto itself. Finding such invariant subspaces is useful. (Google, eigenfaces).

Here is how $(*)$ can be solved. Clearly $x=0$ works. But we want nontrivial solutions.

$$Ax = \lambda x$$

$$Ax - \lambda x = 0$$

$$Ax - \lambda I x = 0$$

$$(A - \lambda I)x = 0 \qquad (\#)$$

You only get nontrivial solutions if $\det(A - \lambda I) = 0$. This gives a polynomial in λ. Solution for λ. There may be several, $\lambda_1 \cdots \lambda_k$, For each solve $(A - \lambda_i I)x = 0$.

Example $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$. Find the eigenvalue. For each eigenvalue find an eigenvector.

Solution $A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix}$

$$\det(A - \lambda I) = (1-\lambda)(-1-\lambda) - 3$$
$$= \lambda^2 + \lambda - \lambda - 1 - 3$$
$$= \lambda^2 - 4.$$

Solve $\lambda^2 - 4 = 0$. $\lambda = \pm 2$. These are the eigenvalues.

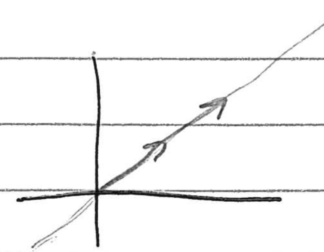
$\boxed{\lambda = 2}$

$$A - 2I = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \left.\begin{array}{l} -x + y = 0 \\ 3x - 3y = 0 \end{array}\right\} \text{redundant.}$$

$x = y$. Pick one, say $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

check: $\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.



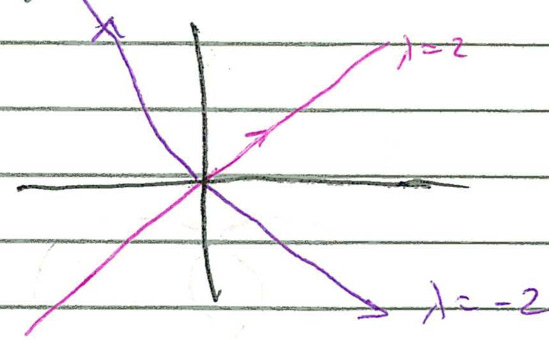
Let $V = \left\{ r\begin{bmatrix} 1 \\ 1 \end{bmatrix} \,\middle|\, r \in \mathbb{R} \right\}$. (called span of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$)

The any vector in V is an eigenvector for $\lambda = 2$.
It is called the eigenspace with eigenvalue 2.

$\boxed{\lambda = -2}$ $A - (-2I) = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$3x + y = 0$. The vector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ works.

The span of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is the eigenspace for $\lambda = -2$.



Example $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$. Find eigenvalues, etc.

$$\begin{vmatrix} 1-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 2 = 0$$

$$\lambda^2 - 3\lambda + 2 - 2 = 0$$

$$(\lambda - 3)\lambda = 0$$

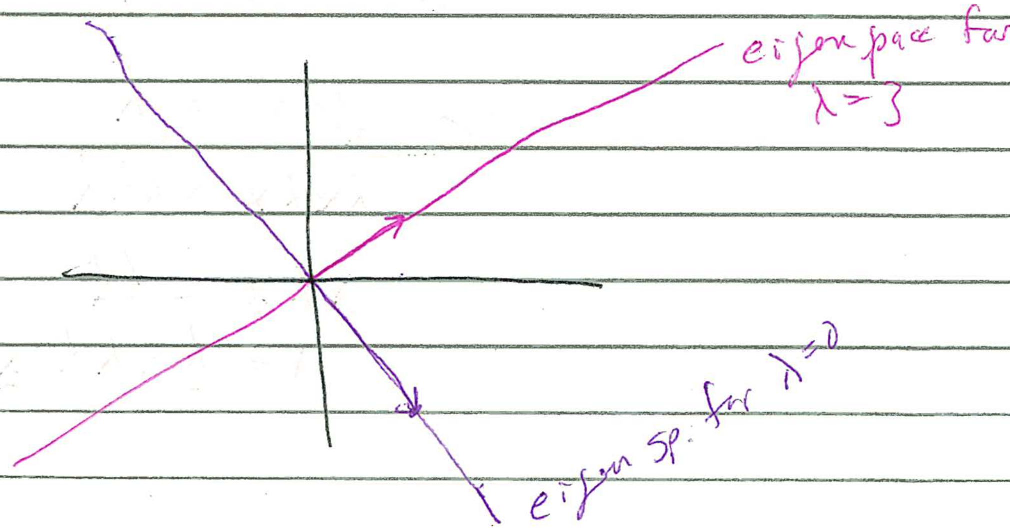
$$\lambda = 3, 0.$$

$\boxed{\lambda = 0}$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x + 2y = 0 \qquad \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\boxed{\lambda = 3}$ $\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$x - y = 0 \qquad x = y \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$



<u>Example</u> Find eigenvalues, vectors, spaces of $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$.

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 0 & -1-\lambda \end{bmatrix} = (2-\lambda)(-1-\lambda) - 0$$

$\lambda = 2$ $\lambda = -1$, are the eigenvalues.

$\boxed{\lambda = 2}$ $A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

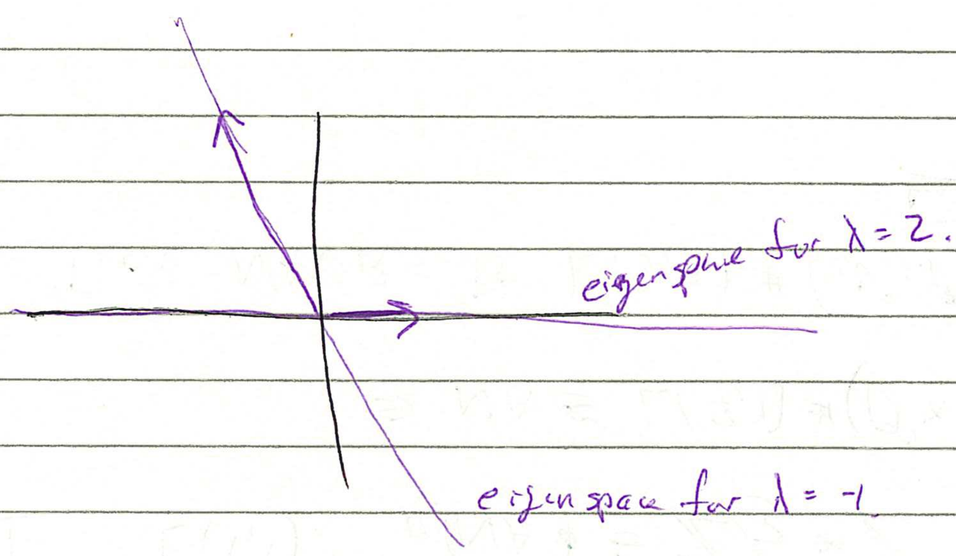
$y = 0$ x is free

use $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as eigenvector.

Eigenspace for $\lambda = 2$ is span of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ = x-axis.

$\boxed{\lambda = -1}$ $A - (-1)I = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$3x + y = 0$. use $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$.



Example $A = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$. Find eigenvalues, etc.

$$\begin{vmatrix} 1-\lambda & -3 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) + 3 = \lambda^2 - 3\lambda + 5 = 0$$

$$\lambda = \frac{3 \pm \sqrt{9 - 4 \cdot 5}}{2}$$

$$= \frac{3 \pm \sqrt{11}\, i}{2}$$

$$\boxed{\lambda = \frac{3}{2} + \frac{\sqrt{11}}{2} i}$$

$$\begin{bmatrix} -\frac{1}{2} - \frac{\sqrt{11}}{2} i & -3 \\ 1 & +\frac{1}{2} - \frac{\sqrt{11}}{2} i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

These are redundant. If you mult. $[1 \quad \frac{1}{2} - \frac{\sqrt{11}}{2} i]$ by $(-\frac{1}{2} - \frac{\sqrt{11}}{2} i)$ you will get $[-\frac{1}{2} - \frac{\sqrt{11}}{2} i \quad -3]$.

Use bottom row. $x + (\frac{1}{2} - \frac{\sqrt{11}}{2} i) y = 0.$

If $y = 1$, $x = -\frac{1}{2} + \frac{\sqrt{11}}{2} i$. So, $\begin{bmatrix} -\frac{1}{2} + \frac{\sqrt{11}}{2} i \\ 1 \end{bmatrix}$

is an eigenvector.

$$\boxed{\lambda = \frac{3}{2} - \frac{\sqrt{11}}{2} i}$$

$$\begin{bmatrix} -\frac{1}{2} + \frac{\sqrt{11}}{2} i & -3 \\ 1 & \frac{1}{2} + \frac{\sqrt{11}}{2} i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} -\frac{1}{2} - \frac{\sqrt{11}}{2} i \\ 1 \end{bmatrix}$ is an eigenvector.

Geometric meaning is different. Rotation!

<u>Example</u> $A = \begin{bmatrix} 4 & -4 & -6 \\ 4 & 14 & 12 \\ -3 & -6 & -3 \end{bmatrix}$. Find eigenvalues. Find a basis for each eigenspace.

You can check that $\det(A - \lambda I) = -(\lambda-6)^2(\lambda-3)$.

Thus the eigenvalue are 3 and 6 with multiplicity 2.

$\boxed{\lambda=3}$ We need to solve $\begin{bmatrix} 1 & -4 & -6 \\ 4 & 11 & 12 \\ -3 & -6 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

But we did this in example X !

$\left\{ \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix} \right\}$ is a basis for the eigenspace.

$\boxed{\lambda=6}$ We need to solve $\begin{bmatrix} -2 & -4 & -6 \\ 4 & 8 & 12 \\ -3 & -6 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

We did this in example Y.

$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for this eigenspace.

$\lambda = 3, \quad \lambda = 6$, mult. 2.

$\boxed{\lambda = 3}$ $\quad (A - 3I) = \begin{bmatrix} 1 & -4 & -6 \\ 4 & 11 & 12 \\ -3 & -6 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{matrix} 1 & -4 & -6 \\ 0 & 27 & 36 \\ 0 & -18 & -24 \end{matrix}$$

$$\begin{matrix} 1 & -4 & -6 \\ 0 & 3 & 4 \\ \cancel{0} & \cancel{3} & \cancel{4} \end{matrix} \qquad \begin{matrix} 1 & -1 & -2 \\ & 3 & 4 \end{matrix}$$

$$\begin{matrix} 1 & -1 & -2 \\ 0 & 1 & 4/3 \end{matrix}$$

$$x - 4y - 6z = 0$$
$$3y + 4z = 0$$

$$\begin{matrix} 1 & 0 & -2/3 \\ 0 & 1 & +4/3 \end{matrix}$$

$$\begin{aligned} x &= \tfrac{2}{3} z \\ y &= -\tfrac{4}{3} z \\ z &= z \end{aligned} \quad \cdot \quad \begin{bmatrix} 2/3 \\ -4/3 \\ 1 \end{bmatrix} z$$

$\begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$ eigenvector.

$A - 6I =$

$\boxed{\lambda = 6}$

$$\begin{bmatrix} -2 & -4 & -6 \\ 4 & 8 & 12 \\ -3 & -6 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x + 2y + 3z = 0$$

$$\begin{aligned} x &= -2y - 3z \\ y &= y \\ z &= z \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} z$$

eigenspace spanned by $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$