Section 7.3
More Linear Algebra

We consider

$$\begin{aligned} b_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, \\ b_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n, \\ &\vdots \\ b_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n, \end{aligned}$$

where the b_i's and a_{ij}'s are given and the x_i's are unknowns. We can also write this as

$$\mathbf{b} = A\mathbf{x}.$$

If **b** is the zero vector the system is said to be **homogeneous**. If not, it is **nonhomogeneous**.

Fact. If $\det A \neq 0$, then A^{-1} exists, and $\mathbf{x} = A^{-1}\mathbf{b}$ is the unique solution.

Fact. If $\det A = 0$, then

- $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, while
- $A\mathbf{x} = \mathbf{b} \neq \mathbf{0}$ has either no solutions or infinitely many.

Examples will come shortly.

Vector Spaces. (This is not in your textbook.)

Let V be a nonempty subset of $\mathbb{R}^n$ or $\mathbb{C}^n$. Then V is called a **vector space** if the following **closure axioms** hold true.

(1) If $\mathbf{v}, \mathbf{w} \in V$, then $\mathbf{v} + \mathbf{w} \in V$.
(2) If $\mathbf{v} \in V$ and $\alpha \in \mathbb{R}$ or $\mathbb{C}$, then $\alpha\mathbf{v} \in V$.

Example. In $\mathbb{R}^2$ lines going through the origin are vector spaces. So are $\{(0,0)\}$ and $\mathbb{R}^2$ itself. In fact these are the only vector spaces in $\mathbb{R}^2$.

Example. In $\mathbb{R}^3$ lines and planes going through the origin are vector spaces. So are $\{(0,0,0)\}$ and $\mathbb{R}^3$ itself. In fact these are the only vector spaces in $\mathbb{R}^3$.

If F is a nonempty set of functions from $\mathbb{R}$ to $\mathbb{R}$, then F is said to be a vector space if the two closure axioms hold true.

Example. The solution set of $y'' + y = 0$ is the vector space $\{C_1 \sin t + C_2 \cos t \,|\, C_1, C_2 \in \mathbb{R}\}$. Think about that.

Linear Dependence and Independence.

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is said to be **linearly independent** if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_k\mathbf{v}_k = \mathbf{0} \qquad (*)$$

has only one solution $c_1 = c_2 = \cdots = c_k = 0$. If there is a nontrivial solution, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is **linearly dependent**.

We can write (*) as a matrix equation

$$A\mathbf{c} = \mathbf{0},$$

where we regard the $\mathbf{v}_i$'s as columns of A and $\mathbf{c} = \begin{bmatrix} c_1 & c_2 & \cdots & c_k \end{bmatrix}^T$. Check this.

Fact. If A is $n \times n$ then the columns of A are linearly independent iff $\det A \neq 0$. The same is true for the rows of A.

Example. Solve $\begin{array}{rcl} 2x - y &=& 4 \\ x + 3y &=& 7 \end{array}$ or show there are no solutions.

Clearly, you can do this any number of ways. The point here is to do it systematically and think about it. We set up an **augmented matrix** and use row operations to put it into **reduced row echelon form** (RREF).

$$\begin{array}{rr|r} 2 & -1 & 4 \\ 1 & 3 & 7 \end{array}$$

$$\begin{array}{rr|r} 0 & -7 & -10 \\ 1 & 3 & 7 \end{array}$$

$$\begin{array}{rr|r} 1 & 3 & 7 \\ 0 & -7 & -10 \end{array}$$

$$\begin{array}{rr|r} 1 & 3 & 7 \\ 0 & 1 & 10/7 \end{array}$$

$$\begin{array}{rr|r} 1 & 0 & 19/7 \\ 0 & 1 & 10/7 \end{array}$$

The 1's on the diagonal are called **pivots**. We see that the unique solution is

$$x = 19/7 \qquad y = 10/7.$$

Example. $\begin{array}{rcl} 2x + 3y &=& 2 \\ 4x + 6y &=& 3 \end{array}$ has no solutions.

Example. $\begin{array}{rcl} 2x + 3y &=& 0 \\ 4x + 6y &=& 0 \end{array}$ has infinitely many solutions. The solution set is the line $y = -2x/3$, which is a vector space.

Example. $\begin{array}{rcl} 2x + 3y &=& 2 \\ 4x + 6y &=& 4 \end{array}$ has infinitely many solutions. The solution set is the line $y = (2 - 2x)/3$, which is not a vector space.

Example X. (Will be used later.) Find all solutions to

$$\begin{bmatrix} 1 & -4 & -6 \\ 4 & 11 & 12 \\ -3 & -6 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

When doing the row operations I won't bother recording the last column of zeros since they will not change value.

$$\begin{array}{rrrr}
1 & -4 & -6 & \\
4 & 11 & 12 & \\
-3 & -6 & -6 & \\
 & & & \\
1 & -4 & -6 & \\
0 & 27 & 36 & R2 - 4R1 \\
0 & -18 & -24 & R3 + 3R1 \\
 & & & \\
1 & -4 & -6 & \\
0 & 3 & 4 & R2 \div 4 \\
0 & 3 & 4 & R3 \div -6 \\
 & & & \\
1 & -4 & -6 & \\
0 & 1 & 4/3 & R2 \div 3 \\
0 & 0 & 0 & R3 - R2 \\
 & & & \\
1 & 0 & -2/3 & R1 + 4R2 \\
0 & 1 & 4/3 & \\
0 & 0 & 0 &
\end{array}$$

This is the RREF. There are only two pivots. This means z will be a **free variable**. We write

$$x = 2z/3$$
$$y = -4z/3$$
$$z = z$$

Or, in matrix form

$$\begin{bmatrix} x \\ y \\ x \end{bmatrix} = \begin{bmatrix} 2/3 \\ -4/3 \\ 1 \end{bmatrix} z.$$

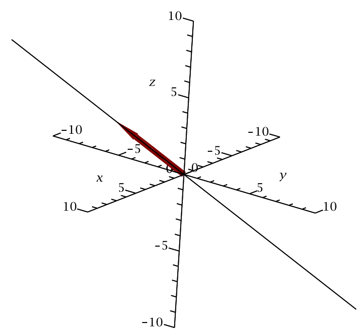
We think of the solution set as the vector space given by the

$$\text{span of } \begin{bmatrix} 2/3 \\ -4/3 \\ 1 \end{bmatrix} = \left\{ \begin{bmatrix} 2/3 \\ -4/3 \\ 1 \end{bmatrix} z : z \in \mathbb{R} \right\}.$$

(It is also called the **nullspace** of the original matrix.) Notice is it equal to

$$\text{span of } \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}.$$

(This is an alternative basis for the solution space.) Below we show this vector and the solution space it spans.



Example Y. (Will be used later.) Find all solutions to

$$\begin{bmatrix} -2 & -4 & -6 \\ 4 & 8 & 12 \\ -3 & -6 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{array}{cccr} -2 & -4 & -6 & \\ 4 & 8 & 12 & \\ -3 & -6 & -9 & \end{array}$$

$$\begin{array}{cccr} 1 & 2 & 3 & R1 \div -2 \\ 1 & 2 & 3 & R2 \div 4 \\ 1 & 2 & 3 & R3 \div -3 \end{array}$$

$$\begin{array}{cccr} 1 & 2 & 3 & \\ 0 & 0 & 0 & R2 - R1 \\ 0 & 0 & 0 & R3 - R1 \end{array}$$

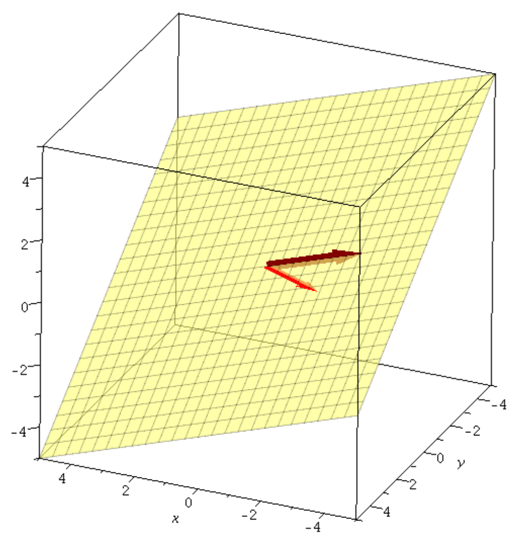
There is only one pivot. We have $x+2y+3z=0$. This gives a plane in $\mathbb{R}^3$. It goes through the origin and is hence a vector space. We can rewrite this as follows.

$$\begin{array}{rcrcr} x & = & -2y & - & 3z \\ y & = & y & & \\ z & = & & & z \end{array}$$

We can write this is matrix form as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} z.$$

Thus, the solution set is the set of all vector tat can be written as a linear combination of these two vectors. This is called the span of $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$. A graph of this plane is shown below.



Eigenvalues, Eigenvectors and Eigenspaces

Eigen is the German word for characteristic. Consider the problem

$$A\mathbf{x} = \lambda\mathbf{x}$$

where A is an $n\ xn$ given matrix of real constants, $\mathbf{x}$ is a $1 \times n$ column vector of unknowns and λ is an unknown scalar (real or complex). We want to find values for λ and vectors for $\mathbf{x}$. Such a λ is called an **eigenvalue** for A and a corresponding (nonzero) vector $\mathbf{x}$ is called an **eigenvector** for λ and A. The set of all eigenvectors for λ (including now the zero vector) is a vector space and is called the **eigenspace** for λ and A.

Geometrically, if λ and $\mathbf{x}$ are real, the matrix A, as a map from $\mathbb{R}^n$ to $\mathbb{R}^n$, maps the subspace $\{r\mathbf{x} : r \in \mathbb{R}\}$ to itself. (Onto if $\lambda \neq 0$.)

Eigenvectors come up in surprising places. When you do a Google search the list you get is actually an eigenvector for a large linking matrix. See https://en.wikipedia.org/wiki/PageRank. Facial recognition programs use eigenfaces, which are in fact eigenvectors. See https://en.wikipedia.org/wiki/Eigenface.

Now, here is how we can solve $A\mathbf{x} = \lambda\mathbf{x}$. As stated we want nonzero solutions for $\mathbf{x}$.

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ A\mathbf{x} - \lambda\mathbf{x} &= \mathbf{0} \\ A\mathbf{x} - \lambda I\mathbf{x} &= \mathbf{0} \\ (A - \lambda I)\mathbf{x} &= \mathbf{0} \end{aligned}$$

But we can only get nontrivial solutions for $\mathbf{x}$ if $\det(A - \lambda I) = 0$. Now $\det(A - \lambda I)$ is just a polynomial in λ. We find its roots and then for each go back and solve for $\mathbf{x}$.

The rest of these notes are examples.

Example. Let $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$. Find the eigenvalues and for each find an eigenvector.

Solution. $A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix}$.

$$\begin{aligned} \det(A - \lambda I) &= (1-\lambda)(-1-\lambda) - 3 \\ &= \lambda^2 + \lambda - \lambda - 1 - 3 \\ &= \lambda^2 - 4. \end{aligned}$$

Solving $\lambda^2 - 4 = 0$ gives $\lambda = \pm 2$.

Suppose $\lambda = 2$.

Then $A - 2I = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}$. We solve $\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We only need one nontrivial solution. The two rows are redundant. Both imply $x = y$. Thus, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ will work, as would any nonzero multiple of it. The span of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigenspace for $\lambda = 2$.

Suppose $\lambda = -2$.

Now we have $A - (-2)I = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$. a solution for $\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. The eigenspace is its span.

Example. Let $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$. Find the eigenvalues and for each find an eigenvector.

Solution. $A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 0 & -1-\lambda \end{bmatrix}$. The determinate is $(2 - \lambda)(-1 - \lambda)$. Thus, the eigenvalues are $\lambda = 2$ and -1.

Suppose $\lambda = 2$. Then $A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}$. We need a solution to $\begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Notice we must have $y = 0$, but x can take any value. We can use $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for an eigenvector. The eigenspace is just the x-axis.

Suppose $\lambda = -1$. Now $A + I = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$. A solution to $\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$. The span of this eigenvector is the eigenspace for $\lambda = -1$.

Example. Let $A = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$. Find the eigenvalues and for each find an eigenvector.

Solution. $\det \begin{bmatrix} 1-\lambda & -3 \\ 1 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda)+3 = \lambda^2 - 3\lambda + 5.$ The roots are

$$\lambda = \frac{3 \pm \sqrt{9-4\cdot 5}}{2} = \frac{3 \pm i\sqrt{11}}{2}.$$

So, they are complex. The eigenvectors will be also.

Suppose $\lambda = \frac{3}{2} + \frac{\sqrt{11}}{2}i.$ Then we have to solve

$$\begin{bmatrix} -\frac{1}{2} - \frac{\sqrt{11}}{2}i & -3 \\ 1 & \frac{1}{2} - \frac{\sqrt{11}}{2}i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It may not be obvious that the two equations this represents are redundant. But, if you multiply the second row by $-\frac{1}{2} - \frac{\sqrt{11}}{2}i$ it will turn into the first row. Check this. From the bottom row we get

$$x + \left(\frac{1}{2} - \frac{\sqrt{11}}{2}i\right) y = 0.$$

Thus, we can use $\begin{bmatrix} \frac{1}{2} - \frac{\sqrt{11}}{2}i \\ -1 \end{bmatrix}$ as an eigenvector.

Suppose $\lambda = \frac{3}{2} - \frac{\sqrt{11}}{2}i.$ I'll let you check the details, but the result is exactly the complex conjugate of the first case.

The geometric meaning is not obvious. It turns out complex eigenvalues are related to rotation.

Our Final Example. Let $A = \begin{bmatrix} 4 & -4 & -6 \\ 4 & 14 & 12 \\ -3 & -6 & -3 \end{bmatrix}$. Find the eigenvalues and a basis for each eigenspace.

Solution. You can check that $\det(A - \lambda I) = -(\lambda-3)(\lambda-6)^2$. Thus the eigenvalues are 3 and 6 with multiplicity 2.

Suppose $\lambda = 3.$ We need to solve $\begin{bmatrix} 1 & -4 & -6 \\ 4 & 11 & 12 \\ -3 & -6 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

But, we did this in Example X! Thus $\left\{ \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix} \right\}$ is a basis for the eigenspace for $\lambda = 3$.

$\boxed{\text{Suppose } \lambda = 6.}$ We need to solve $\begin{bmatrix} -2 & -4 & -6 \\ 4 & 8 & 12 \\ -3 & -6 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

We did this in Example Y. Thus, $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for this two dimensional eigenspace.