7.4 Basics of $X' = AX$.

Thm 7.4.1 Suppose $X_1(t), X_2(t), \ldots, X_k(t)$ are $n \times 1$ column vectors that are solutions to $X' = AX$. (A is $n \times n$ and can be constants or functions.) Then

$$X = c_1 X_1 + c_2 X_2 + \cdots + c_k X_k$$

is also an solution where c_i are real or complex constants.

Pf Just plug in. $X' = c_1 X_1' + c_2 X_2' + \cdots + c_k X_k'$

$$= c_1 A X_1 + c_2 A X_2 + \cdots + c_k A X_k$$

$$= A(c_1 X_1 + \cdots + c_k X_k) = AX. \quad \square$$

Thm 7.1.2 The initial value problem $X' = AX$, $X(t_0) = b$, some give column vector of numbers, has a unique solution. If the entries of A are constants this solution is valid $\forall t \in \mathbb{R}$. If the entries are functions that are cont. on $t \in (\alpha, \beta)$, then so is the solution.

Pf MATH 505.

Thm 7.4.2 Suppose $x_1(t), \ldots, x_n(t)$ are Linearly Independent solutions of $X' = AX$, where A is $n \times n$. Let $\phi(t)$ be any other solution. Then there exist unique numbers s.t.

$$\phi = c_1 X_1 + c_2 X_2 + \cdots c_n X_n.$$

Pf Pick $t_0 \in \mathbb{R}$ where the solutions are valid. Let $b = \phi(t_0)$. Let $X = [x_1(t_0) \cdots x_n(t_0)]$. Let $c = X^{-1}b$.

Def The set $\{x_1 \ldots x_n\}$ is called a <u>fundamental set of solutions</u>. It is a basis for the solution space. The matrix X is called a <u>fundamental matrix</u>.

A special case is when x_1 satisfies $[1\ 0 \cdots 0]^T$,
x_2 " $[0\ 1\ 0 \cdots 0]^T$
etc.

Then X is the <u>special fundamental matrix</u>. (See §7.7)

Thm 7.4.5 If $x' = Ax$ and $x = u + iv$, where A, u, v have real valued entries, then $u' = Au$ and $v' = Av$.

Pf
$$x' - Ax = 0 \Rightarrow (u + iv)' - A(u + iv) = 0$$
$$\Rightarrow (u' - Au) + i(v' - Av) = 0$$
$$\Rightarrow u' = Au \text{ and } v' = Av. \qquad \square$$

Note: In 7.9 we will look at $x' = Ax + g(t)$.