On the Wronskian

In MATH 305 you used the Wronskian to determine if solutions were L.I. Here the Wronskian takes a slightly different form, but it plays the same role.

Ex (from Section 3.2): Show e^t and e^{2t} are L.I.

$$W(e^t, e^{2t}) = \begin{vmatrix} e^t & e^{2t} \\ (e^t)' & (e^{2t})' \end{vmatrix} = 2e^{3t} - e^{3t} = e^{3t} \neq 0$$

Hence e^t and e^{2t} are L.I.

Ex ~~Show~~ (Section 3.2) Show $\sin t$, $\cos t$ are L.I.

$$W(\sin t, \cos t) = \begin{vmatrix} \sin t & \cos t \\ (\sin t)' & (\cos t)' \end{vmatrix} = -\sin^2 t - \cos^2 t = -1 \neq 0.$$

Hence $\sin t$ and $\cos t$ are L.I.

The Wronskian of 3 functions was $W(f,g,h) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}$. etc

See Section 4.1.

In Ch. 7 we are dealing with vector valued functions. If $v_1(t)$ and $v_2(t)$ are L.I. at t_0, then

$$W(v_1(t_0), v_2(t_0)) = \det[v_1(t_0)\ v_2(t_0)] \neq 0.$$

Ex 1 from my 7.1 lecture notes solved
$$x' = 2x - 3y$$
$$y' = x - 2y.$$

We get
$$x = 3c_1 e^{t} + c_2 e^{-t}$$
$$y = c_1 e^{t} + c_2 e^{-t}$$

Suppose, $x(0) = 1$ $y(0) = 0$. Then
$$3c_1 + c_2 = 1$$
$$c_1 + c_2 = 0 \qquad c_2 = -c_1$$
$$\Rightarrow 2c_1 = 1$$
$$c_1 = \tfrac{1}{2} \qquad c_2 = -\tfrac{1}{2}.$$

Suppose $x(0) = 0$, $y(0) = 1$. Then
$$3c_1 + c_2 = 0$$
$$c_1 + c_2 = 1$$
$$2c_1 = -1$$
$$c_1 = -\tfrac{1}{2} \qquad c_2 = \tfrac{3}{2}.$$

Now we have a pair of solutions:
$$V_1(t) = \begin{bmatrix} \frac{3}{2}e^{t} + \frac{1}{2}e^{-t} \\ \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} \end{bmatrix} \qquad V_2(t) = \begin{bmatrix} -\frac{3}{2}e^{t} + \frac{3}{2}e^{-t} \\ -\frac{1}{2}e^{t} + \frac{3}{2}e^{-t} \end{bmatrix}.$$

We can use the Wronskian to show these are L.I.
By Thm 7.4.2 we can solve any init. value problem with linear combinations of V_1 and V_2.

$$W(V_1(t), V_2(t)) = \begin{vmatrix} \frac{3}{2}e^{t} - \frac{1}{2}e^{-t} & -\frac{3}{2}e^{t} + \frac{3}{2}e^{-t} \\ \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} & -\frac{1}{2}e^{t} + \frac{3}{2}e^{-t} \end{vmatrix}$$

Before we do this let's look at one more theorem.

Thm 7.4.3 If $v_1(t), \dots v_n(t)$ are solutions to $v' = Av$ on some interval (α, β), then for $t \in (\alpha, \beta)$, $W(v_1(t), \dots, v_n(t))$ is always zero or never zero.

For proofs see Problems 2, 8, 9 in this section.

Back to our example. Let $t=0$.

$$W(v_1(0), v_2(0)) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0. \quad \text{Done.}$$