7.5 We show how to find solutions to

$$x' = Ax \qquad (\star)$$

when A is an $n \times n$ matrix of real constants and has n real distinct eigenvalues.

Recall that equations like $y' = ay$ had solutions $y = Ce^{at}$. We try this for $x' = Ax$. Let $x = ve^{rt}$

Where v is a column vector of ~~c~~ unknown constants. Suppose

$$x' = Ax.$$

Now $x' = vre^{rt}$. So we have

$$vre^{rt} = Ave^{rt}$$

or

$$Av = rv.$$

Hence r must be an eigenvalue with eigenvector v.

Thus, to solve $(\star)$ we find the eigenvectors and eigenvalues of A.

Example $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -16 & -6 \\ 45 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} -16-\lambda & -6 \\ 45 & 17-\lambda \end{bmatrix}$ $\quad \det = \lambda^2 - \lambda - (16)(17) + 270$

$$= \lambda^2 - \lambda - 2$$
$$= (\lambda+1)(\lambda-2)$$
$$\lambda = -1, 2$$

$\boxed{\text{Let } \lambda = -1}$

$\begin{bmatrix} -15 & -6 \\ 45 & 18 \end{bmatrix}$ $\quad 15x_1 + 6x_2 = 0$

$$5x_1 + 2x_2 = 0$$
$$x_1 = \frac{-2}{5} x_2 \qquad \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

Hence $X = \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-t}$ is a solution.

$\boxed{\text{Let } \lambda = 2}$

$\begin{bmatrix} -18 & -6 \\ 45 & 15 \end{bmatrix}$ $\quad 3x_1 + x_2 = 0$

$$x_1 = -\frac{1}{3} x_2 \qquad \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Hence $X = \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t}$ is a solution.

It is easy to check $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ are L.I.

Hence nonzero multiples are L.I. So, the general solution is

$$X = C_1 \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t}.$$

Suppose we had initial conditions : $x_1(0)=1$, $x_2(0)=2$.
Find C_1 & C_2.

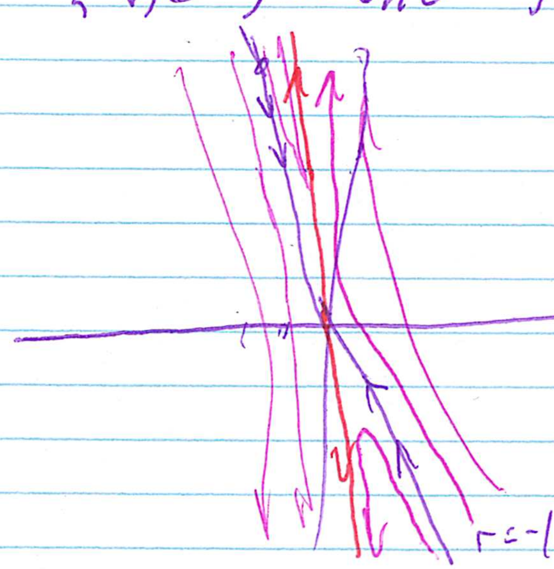
$$X(0) = C_1\begin{bmatrix}-2\\5\end{bmatrix}e^0 + C_2\begin{bmatrix}-1\\3\end{bmatrix}e^0 = \begin{bmatrix}-2 & -1\\5 & 3\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

$$\begin{bmatrix}-2 & -1\\5 & 3\end{bmatrix}^{-1} = \begin{bmatrix}-3 & -1\\ 5 & +2\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}-3 & -1\\5 & 2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-5\\9\end{bmatrix}$$

$$x = -5\begin{bmatrix}-2\\5\end{bmatrix}e^{-t} + 9\begin{bmatrix}-1\\3\end{bmatrix}e^{2t} = \begin{bmatrix}10e^{-t} - 9e^{2t}\\ -25e^{-t} + 27e^{2t}\end{bmatrix}.$$

<u>Fact</u> If A is $n\times n$ real matrix has n real distinct e.values, then the eigenvectors are LI.
If $r_1, \dots r_n$ are the eigenvalues and if $v_1, \dots v_n$ are the corresp. e.vectors then $\{v_1 e^{r_1 t}, \dots v_n e^{r_n t}\}$ is a fund. sol. set.



Example using Maple

$$x' = -16x-6y$$

$$y' = 45x+17y$$

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> with(DEtools):with(plots):
> df:=dfieldplot([D(x)(t)=-16*x(t)-6*y(t),  D(y)(t)=45*x(t)+17*y(t)
  ], [x(t),y(t)],t=0..1,x=-10..10,y=-10..10, color=black,
  thickness=2):
> es1:=plot(-5*x/2,x=-10..10,color=red):
> es2:=plot(-3*x,x=-10..10,color=blue):
> display(es1,es2,df,view=[-10..10,-10..10]);
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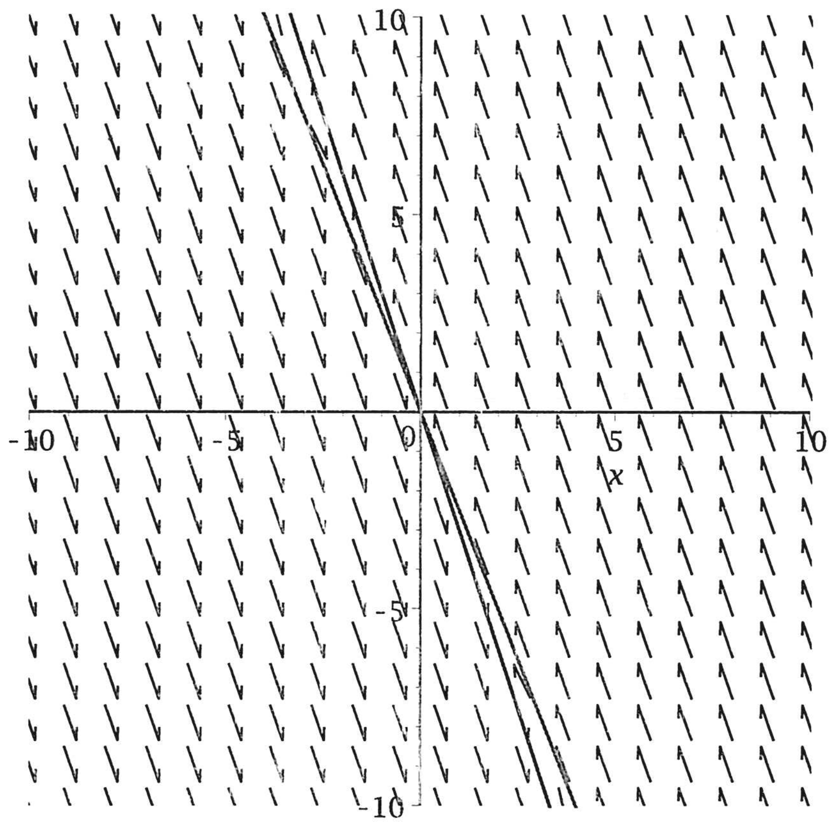


Example with eigenvalue zero.

$$X' = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} X.$$

Solution

$$\begin{vmatrix} 1-\lambda & 2 \\ -1 & -2-\lambda \end{vmatrix} = (\lambda-1)(\lambda+2)+2$$

$$\lambda^2+\lambda-\cancel{2}+\cancel{2}$$

$$\lambda(\lambda+1)$$

$$\lambda = 0, -1.$$

$\boxed{\lambda = 0}$ $\quad x_1 + 2x_2 = 0$

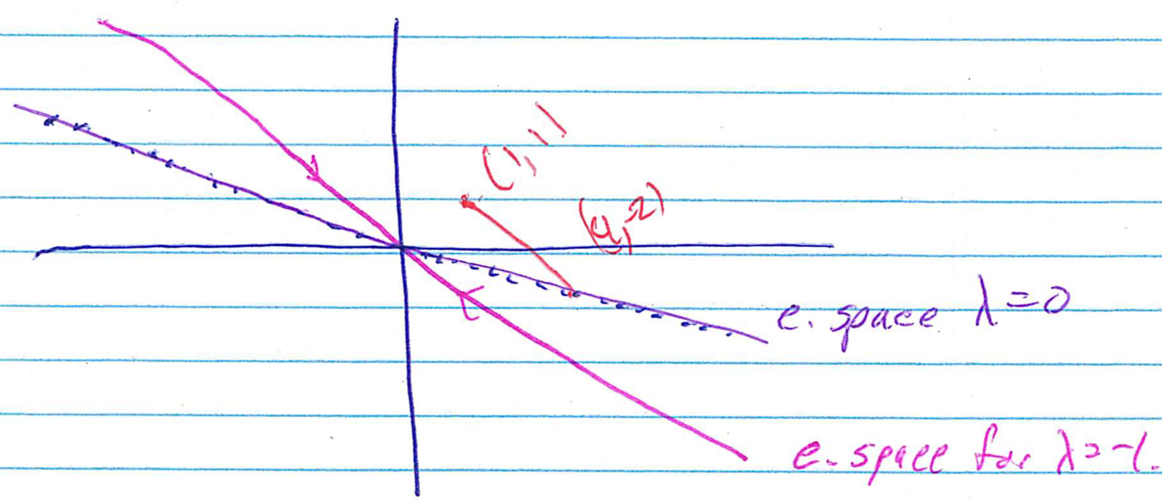
use $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\boxed{\lambda = -1}$ $\quad 2x_1 + 2x_2 = 0$

use $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

General Solution

$$X = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \overset{1}{e^{0}} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} = \begin{bmatrix} -2C_1 + C_2 e^{-t} \\ C_1 + C_2 e^{-t} \end{bmatrix}.$$



Initial Value Problems for this Example.

Suppose, $X(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. $\quad X(0) = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Rightarrow c_1 = 1, \; c_2 = 0.$

Thus $X(t) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, forever!

Every point on the e.space for $\lambda = 0$ is frozen in time.

Suppose $X(0) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$. Then $c_1 = 0, \; c_2 = 2.$

$$X(t) = \begin{bmatrix} 2e^{-t} \\ -2e^{-t} \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Suppose $X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $\quad X(0) = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\begin{matrix} 1 & -1 & 1 \\ 0 & -1 & 3 \end{matrix}$$

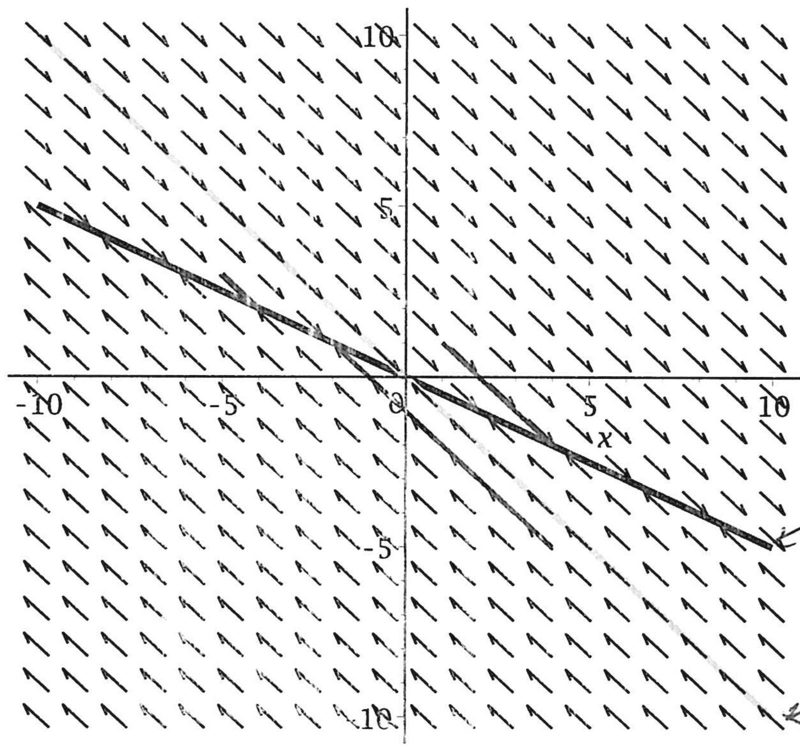
$$\begin{matrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{matrix}$$

$$X(t) = -2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \longrightarrow \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

Example Using Maple

$$x' = x + 2y$$
$$y' = -x - 2y$$

```
> with(DEtools):with(plots):
> vf:=phaseportrait([D(x)(t)=x(t)+2*y(t), D(y)(t)=-x(t)-2*y(t)], [x
  (t),y(t)], t=0..20,[[x(0)=1,y(0)=1],[x(0)=-5,y(0)=3],[x(0)=4,y(0)
  =-5]],x=-10..10,y=-10..10,linecolor=red,color=black,thickness=2):
> esp0:=plot(-x/2,x=-10..10,color=blue,thickness=3):esp1:=plot(-x,
  x=-10..10,color=pink,thickness=3):
> display(esp0,esp1,vf);
```



blue, e-space $\lambda=0$

Pink, espace $\lambda=-1$

Explanation: Pink line is eigenspace for eigenvalue −1. All points flow to (0,0). Blue line is eigenspace for eigenvalue zero. Points on blue line are frozen in time. All other points will flow toward the blue line on a path pararllel to the pink line.

Example: $X' = \begin{bmatrix} -3 & -10 & 2 \\ 4 & 10 & -2 \\ 4 & 8 & -1 \end{bmatrix} X. \quad X(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$

Solution: Find eigenvalues.

$$\begin{vmatrix} -3-\lambda & -10 & 2 \\ 4 & 10-\lambda & -2 \\ 4 & 8 & -1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & -\lambda & 0 \\ 4 & 10-\lambda & -2 \\ 4 & 8 & -1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\big((10-\lambda)(-1-\lambda)+16\big) + \lambda\big(-4-4\lambda+8\big)$$

$$\quad " \quad \big((\lambda-10)(\lambda+1)+16\big) \qquad 4-4\lambda$$

$$4\lambda(1-\lambda)$$

$$= (1-\lambda)\left[\lambda^2-9\lambda-10+16+4\lambda\right]$$

$$= (1-\lambda)\left[\lambda^2-5\lambda+6\right] = (1-\lambda)(\lambda-2)(\lambda-3)$$

The eigenvalues are 1, 2, 3.

Find Eigenvectors

$\boxed{\lambda=1}$ Solve $\begin{bmatrix} -4 & -10 & 2 \\ 4 & 9 & -2 \\ 4 & 8 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 4 & 10 & -2 \\ 0 & -1 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} 4x_1 \quad -2x_3 &= 0 \\ x_2 \quad &= 0 \\ x_3 &= x_3 \end{aligned}$$

$$x_1 = \tfrac{1}{2}x_3 \qquad x = \begin{bmatrix} \tfrac{1}{2} \\ 0 \\ 1 \end{bmatrix} x_3. \quad \text{use } \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$
$$x_2 = 0$$
$$x_3 = x_3$$

$\boxed{\lambda = 2}$ Solve $\begin{bmatrix} -5 & -10 & 2 \\ 4 & 8 & -2 \\ 4 & 8 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{matrix} -1 & -2 & 0 \\ 4 & 8 & -2 \\ 0 & 0 & -5 \end{matrix} \qquad x_3 = 0$$

$$\begin{matrix} 1 & 2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & -5 \end{matrix}$$

$$\begin{matrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} \qquad \begin{aligned} x_1 &= -2x_2 \\ x_2 &= x_2 \\ x_3 &= 0 \end{aligned}$$

$$x = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} x_2 \quad \text{use } \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

$\boxed{\lambda = 3}$ Solve $\begin{bmatrix} -6 & -10 & 2 \\ 4 & 7 & -2 \\ 4 & 8 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$\begin{matrix} 3 & 5 & -1 \\ 4 & 7 & -2 \\ 1 & 2 & -1 \end{matrix}$$

$$\begin{matrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{matrix} \qquad \begin{aligned} x_1 &= -3x_3 \\ x_2 &= 2x_3 \\ x_3 &= x_3 \end{aligned}$$

$X = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} x_3$ use $\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$.

Summary $\lambda = 1$, $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\lambda = 2$ $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ $\lambda = 3$ $\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$

General solution is

$$X = C_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^{t} + C_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} e^{2t} + C_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} e^{3t}$$

Initial Value Problem, $X(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

$$X(0) = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Use row ops

$$\begin{matrix} 1 & -2 & -3 & 1 \\ 0 & 1 & 2 & 0 \\ 2 & 0 & 1 & 1 \end{matrix}$$

$$\begin{matrix} 1 & -2 & -3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 4 & 7 & -2 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & -2 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{matrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

Final Answer.

$$X(t) = \begin{bmatrix} -e^{t} + 8e^{2t} - 6e^{3t} \\ -4e^{2t} + 4e^{3t} \\ -2e^{t} + 2e^{3t} \end{bmatrix}.$$