7.6 Complex Eigenvalues

Recall: $e^{a+ib} = e^{a}\left(\cos(b) + i\sin(b)\right)$.

Facts: Let A be an $n \times n$ real matrix.

- If $\lambda_+ = a+ib$ is an eigenvalue, so is $\lambda_- = a-ib$.
- If $\begin{bmatrix} \alpha_1 + i\beta_1 \\ \vdots \\ \alpha_n + i\beta_n \end{bmatrix}$ is an eigenvector for $\lambda_+ = a+ib$,

  then $\begin{bmatrix} \alpha_1 - i\beta_1 \\ \vdots \\ \alpha_n - i\beta_n \end{bmatrix}$ is an eigenvector for $\lambda_- = a-ib$.

- If a set of fundamental solutions contains a conjugate pair of complex functions, $u \pm iv$, these can be replaced by the real valued functions, u, v. (See Exercise #27.)

<u>Example</u> $v' = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} v$. e.values are $-1 \pm 2i$

E. vector for $-1+2i$:

$$\begin{bmatrix} -1-(-1+2i) & 2 \\ -2 & -1-(-1+2i) \end{bmatrix} = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}$$

Solve: $\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 2i & 0 \\ -2 & -2i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$a + bi = 0$ $\qquad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} b$, choose $b=1$

$b = b$

An e.vector for $-1+2i$ is $\begin{bmatrix} -i \\ 1 \end{bmatrix}$

For $-1-2i$ I got $\begin{bmatrix} i \\ 1 \end{bmatrix}$.

What to do with these??

The general complex solution is

$$v = C_1 \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{(-1+2i)t} + C_2 \begin{bmatrix} i \\ 1 \end{bmatrix} e^{(-1-2i)t}, \quad C_1, C_2 \in \mathbb{C}$$

But to get general real solution write

$$\begin{bmatrix} -i \\ 1 \end{bmatrix} (\underbrace{\cos(2t)}_{c} + i \underbrace{\sin(2t)}_{s}) e^{-t}$$

$$= \begin{bmatrix} -ic + s \\ c + is \end{bmatrix} e^{-t} = \begin{bmatrix} s \\ c \end{bmatrix} e^{-t} + i \begin{bmatrix} -c \\ s \end{bmatrix} e^{-t}$$

Then general real solution is

$$v = C_1 \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -\cos(2t) \\ \sin(2t) \end{bmatrix} e^{-t}, \quad C_1, C_2 \in \mathbb{R}.$$

Now, Suppose $x_1(0) = 1$, $x_2(0) = 1$ are initial values.

$$v(0) = C_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus, $C_1 = 1$ and $C_2 = -1$.

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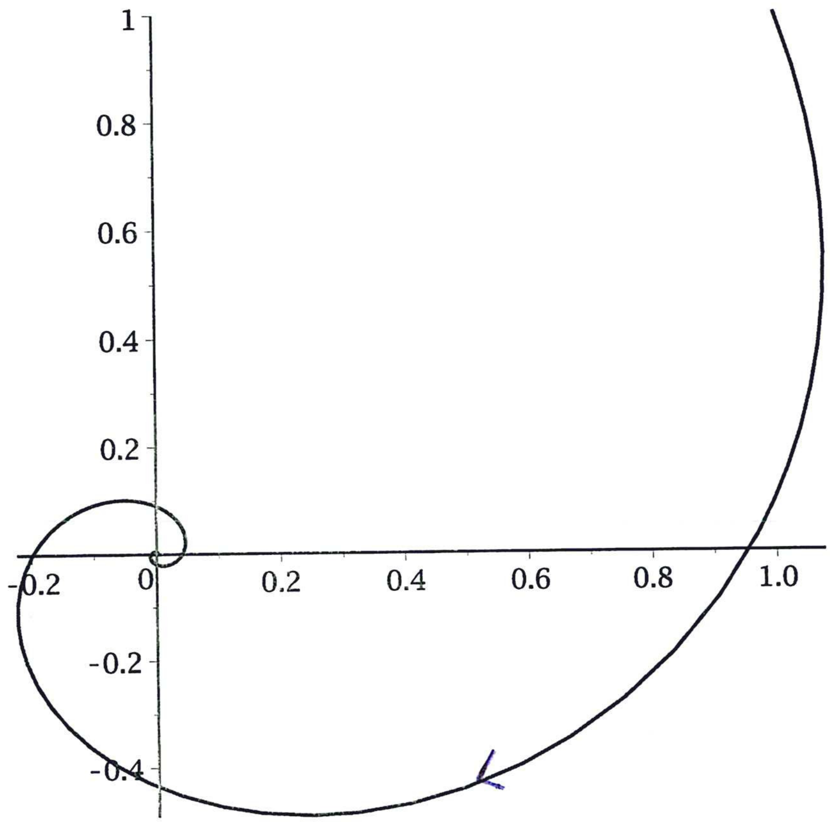
> plot([(cos(2*t)+sin(2*t))*exp(-t),(cos(2*t)-sin(2*t))*exp(-t),t=
0..4*Pi]);
```



Example with pure imaginary eigenvalues

Find the general solution to $X' = \begin{bmatrix} 2 & -8 \\ 1 & -2 \end{bmatrix} X$, and plot some solution curves.

Solution Find eigenvalues.

$$\begin{vmatrix} 2-\lambda & -8 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda-2)(\lambda+2)+8$$
$$= \lambda^2-4+8 = \lambda^2+4. \qquad \lambda = \pm 2i$$

Find eigenvector for $\lambda = 2i$.

$$\begin{bmatrix} 2-2i & -8 \\ 1 & -2-2i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Note These two equations have to be redundant, ie, one is a multiple of the other. To see this mult. row 2 by $2-2i$,
$(2-2i)(-2-2i) = -4(1-i)(1+i) = -8$.

I'll just use the 2^{nd} row: $a - (2+2i)b = 0$.
For an eigenvector use $\begin{bmatrix} 2+2i \\ 1 \end{bmatrix}$.

The general complex solution is

$$C_1 \begin{bmatrix} 2+2i \\ 1 \end{bmatrix} e^{2it} + C_2 \begin{bmatrix} 2-2i \\ 1 \end{bmatrix} e^{-2it}$$

but we don't really need that. To get the general real solution write

$$\begin{bmatrix} 2+2i \\ 1 \end{bmatrix} (\underbrace{\cos(2t)}_{c} + i \underbrace{\sin(2t)}_{s}) = \begin{bmatrix} 2c-2s \\ c \end{bmatrix} + \begin{bmatrix} 2c+2s \\ s \end{bmatrix} i$$

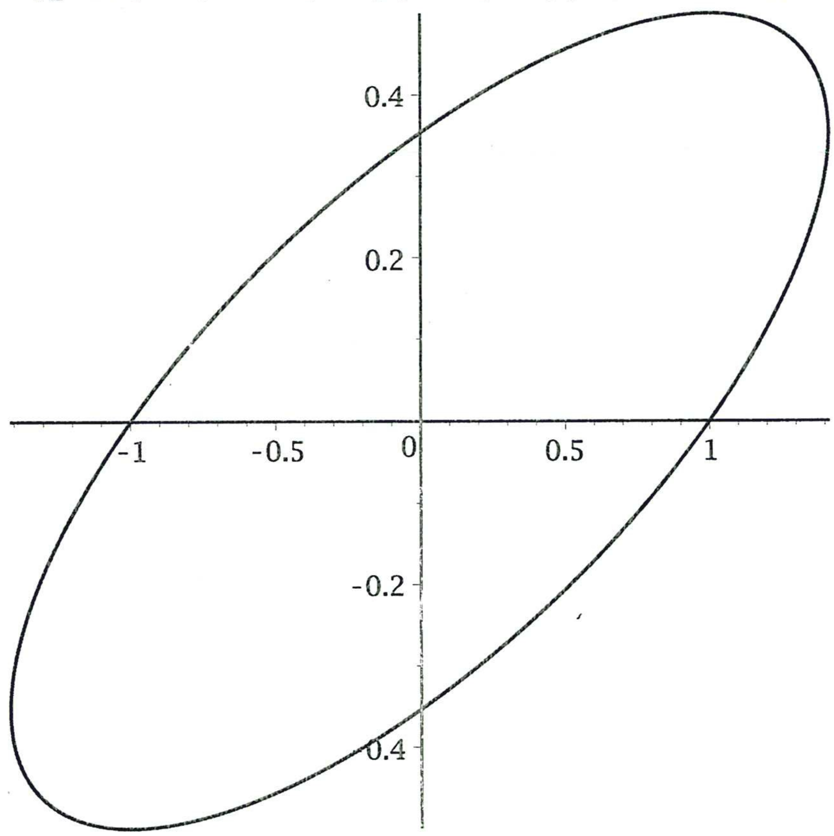
Then the general real sol. in $X = C_1 \begin{bmatrix} 2\cos(2t)-2\sin(2t) \\ \cos(2t) \end{bmatrix} + C_2 \begin{bmatrix} 2c+2s \\ s \end{bmatrix}$

Suppose $X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Now $X(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Thus $c_1 = 0$, $c_2 = \frac{1}{2}$.

We get $x_1(t) = \cos(2t) + \sin(2t)$
$x_2(t) = \frac{1}{2} \sin(2t)$.

```
> plot([cos(2*t)+sin(2*t),sin(2*t)/2,t=0..Pi]);
```



For 2x2 systems with complex eigenvalues

if real part is < 0, inward spiral, $(0,0)$ is attracting
~~asympto~~ asymptotically stable

if real part is > 0, outward spiral, $(0,0)$ is repelling
unstable

if real part $= 0$, get ellipses, $(0,0)$ is called
a center.
stable, but not
asy. stable.