Outline: I. Fund. Matrix and Special Fund. Matrix.
II. $\exp(At)$
III. Diagonalization

7.7 Fundamental Matrices.

Recall Thm 7.1.2 Given $v' = Av$ and $v(t_0) = v_0$ there is a unique solution. [A proof is outlined in Problem 18 in 7.7.]

Thm 7.4.2: Let $\{v_1(t), v_2(t), \dots v_n(t)\}$ be L.I. solutions to $v' = Av$. Then

1. For any v_0 $\exists!$ $c_1, \dots c_n$ s.t. $v = \sum c_i v_i(t)$ solve $v' = Av$, $v(t_0) = v_0$.
2. In other words they form a basis for the solution space.

Def Let $\Psi(t) = [v_1(t) \dots v_n(t)]$ be a matrix with $v_i' = Av_i$ and $\{v_1 \dots v_n\}$ L.I. Then Ψ is called a <u>fundamental matrix</u> for $v' = Av$.

Suppose $v(t_0) = v_0$ and we found $c_1, \dots c_n$ s.t. $v_0 = \sum c_i v_i(t_0)$.
Rewrite as $v_0 = \Psi(t_0) C$, where $C = [c_1 \dots c_n]^T$.
Now
$$v(t) = \sum c_i v_i(t) = \Psi(t) \underset{\Psi^{-1} v_0}{C} = \Psi(t)\, \Psi(t_0)^{-1} v_0$$

Now we define a special fund. matrix, Φ.

Let $e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ ith place. Let $v_i(t)$ be the unique solution to $v' = Av$, $v(t_0) = e_i$ for $i = 1 \dots n$.

Define $\Phi(t) = [v_1(t) \dots v_n(t)]$.

Here is a clever way to compute $\Phi(t)$.

Define $\exp(M) = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots$

Consider $\exp(At)$.

$$\frac{d}{dt}\exp(At) = \frac{d}{dt}\left(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots\right)$$

$$= A + A^2t + \frac{A^3t^2}{2!} + \frac{A^4t^3}{3!} + \cdots$$

$$= A\left(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots\right) = A\exp(At).$$

Thus $\exp(At)$ solves $V' = AV$ (now V is $n \times n$)

and $\exp(A \cdot 0) = I = [e_1\ e_2\ \cdots\ e_n]$.

Thus, $\Phi(t) = \exp(At)$.

But how do we compute $\exp(At)$?

This leads to diagonalization.

Diagonalization

Def Let A and B be $n \times n$ matrices.

If $\exists$ an $n \times n$ invertable matrix T s.t.

$$B = T^{-1} A T$$

then A and B are said to be similar. (This corresponds to a linear coordinate transformation, also called a change of basis. They will have the same eigenvalues.)

Def An $n \times n$ matrix $D = [d_{ij}]$ is a diagonal matrix if $d_{ij} = 0$ for $i \neq j$.

Def An $n \times n$ matrix A is diagonalizable if it is similar to a diagonal matrix.

Ex $\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$ is diagonalizable since

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

But, how do we find T? The columns of T are eigenvectors for distinct eigenvalues. Notice in this example 2 and 3 are eigenvalues with corresponding eigenvectors $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Here is why this works in the 2x2 case. Let A be a 2x2 matrix with eigenvalues λ_1, λ_2, $\lambda_1 \neq \lambda_2$. Suppose v_1 and v_2 are their resp. eigenvectors:

$$Av_1 = \lambda_1 v_1 \qquad Av_2 = \lambda_2 v_2.$$

Let $T = [v_1 \ v_2]$. Let w_1, w_2 be the rows of T^{-1}. Then

$$T^{-1}T = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} [v_1 \ v_2] = \begin{bmatrix} w_1 \cdot v_1 & w_1 \cdot v_2 \\ w_2 \cdot v_1 & w_2 \cdot v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now consider

$$T^{-1}AT = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} [\lambda_1 v_1 \ \ \lambda_2 v_2] = \begin{bmatrix} \lambda_1 w_1 v_1 & \lambda_2 w_1 v_2 \\ \lambda_1 w_2 v_1 & \lambda_2 w_2 v_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

It works the same in the $n \times n$ case.

Not all matrices are diagonalizable. Some with repeated eigenvalues are not. We will deal with this situation in Section 7.8.

Back to Diff Eqs. Suppose D is a diagonal matrix. Then the system $v' = Dv$ is easy to solve, because it is decoupled.

$$\begin{bmatrix} v_1' \\ v_2' \\ \vdots \\ v_n' \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$\Rightarrow v_i' = \lambda_i v_i \Rightarrow v_i(t) = C_i e^{\lambda_i t}$ for any C_i.

In fact $\begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}$ is the special fund. matrix.

We will show this is $\exp(Dt)$.

$$\exp(Dt) = I + Dt + \frac{D^2t^2}{2!} + \frac{D^3t^3}{3!} + \cdots$$

Notice $D^p = \begin{bmatrix} \lambda_1^p & & & \\ & \lambda_2^p & & \\ & & \ddots & \\ & & & \lambda_n^p \end{bmatrix}$

Thus,

$$\exp(Dt) = \begin{bmatrix} 1+\lambda_1 t + \frac{\lambda_1^2 t^2}{2!} + \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1+\lambda_2 t + \frac{\lambda_2^2 t^2}{2!} + \cdots & 0 & \cdots & 0 \\ & & & & \\ & & & & 1+\lambda_n t + \frac{\lambda_n^2 t^2}{2!} + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{bmatrix}.$$

Now, suppose we have $V' = AV$ and A is diagonalizable,

$$D = T^{-1}AT.$$

We will show that $\exp(Dt) = T^{-1}\exp(At)T$.

Thus, when A has n distinct eigenvalues we can find D and thus we know $\exp(Dt)$ and

$$\exp(At) = T\exp(Dt)T^{-1}$$

is the special fund. matrix.

Proof that $\exp(Dt) = T^{-1}\exp(At)T$.

$$\exp(Dt) = \exp(T^{-1}ATt) = I + T^{-1}ATt + \frac{(T^{-1}AT)^2 t^2}{2!} + \frac{(T^{-1}AT)^3 t^3}{3!} + \cdots$$

Now $I = T^{-1}T$, and $(T^{-1}AT)^2 = (T^{-1}AT)(T^{-1}AT) = T^{-1}A^2T$.

In general
$$(T^{-1}AT)^n = (T^{-1}AT)(T^{-1}AT)(T^{-1}AT)\cdots(T^{-1}AT)$$
$$= T^{-1}A^nT$$

Thus
$$\exp(Dt) = T^{-1}T + T^{-1}ATt + \frac{T^{-1}A^2Tt^2}{2!} + \frac{T^{-1}A^3Tt^3}{3!} + \cdots$$
$$= T^{-1}\left(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots\right)T = T^{-1}\exp(At)T$$

as claimed.

Ex Let $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$. (a) Find $\exp(At)$
(b) Solve $v' = Av$ & $v(0) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

Sol Eigenvalues are 2, 5 with resp. eigenvectors

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Thus $T = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ and $T^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$.

$$\exp(At) = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \frac{1}{3}$$

$$= \frac{1}{3} \begin{bmatrix} e^{2t} + 2e^{5t} & -2e^{2t} + 2e^{5t} \\ -e^{2t} + e^{5t} & 2e^{2t} + e^{5t} \end{bmatrix}$$

Notice $\exp(A \cdot 0) = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = I$.

Now for the initial condition $v(0) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

$$\exp(At) \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 8e^{2t} - 2e^{5t} \\ -8e^{2t} - e^{5t} \end{bmatrix}.$$

At $t = 0$ this is $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.