7.8 Repeated Eigenvalues

If you have an eigenvalue of multiplicity m and you can find m linearly independent eigen-vectors for it, you proceed exactly as before.

If you have an eigenvalue of multiplicity 2 and ~~you~~ the eigenspace is only one-dimensional, you use the following: Let r be the eigenvalue, and let v be an eigenvector for it. Then

$$x = v e^{rt}$$

is a solution. To get a second linearly indepent solution let

$$x = v_1 t e^{rt} + v_2 e^{rt}.$$

Plug into $x' = Ax$ and find v_1 and v_2.
It can be shown this gives a L.I. solution.

__Example__ $x' = \begin{pmatrix} -5 & -1 \\ 4 & -1 \end{pmatrix} x.$

__Solution__ The eigenvalues are $-3, -3$. An eigenvector is $v = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

__Trick__ Let $x = v_1 t e^{-3t} + v_2 e^{-3t}$. Plug into $x' = Ax$.

$$\begin{aligned} x' &= -3 v_1 t e^{-3t} + v_1 e^{-3t} - 3 v_2 e^{-3t} \\ &= -3 v_1 t e^{-3t} + (v_1 - 3 v_2) e^{-3t} \end{aligned}$$

$$Ax = A v_1 t e^{-3t} + A v_2 e^{-3t}$$

Thus,

• $Av_1 = -3v_1 \Rightarrow v_1$ is an eigenvector for -3. Let $v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.
(This always happens.)

• $Av_2 = v_1 - 3v_2 \Rightarrow Av_2 + 3Iv_2 = v_1$

$$\Rightarrow (A + 3I)v_2 = v_1.$$

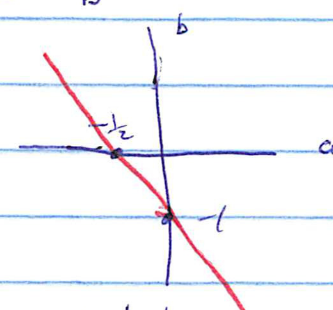
Since, we know v_1 we can solve this.

$$\begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$-2a - b = 1 \qquad a = -\tfrac{1}{2}b - \tfrac{1}{2}$$
$$b = b \qquad b = b$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} b + \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}$$



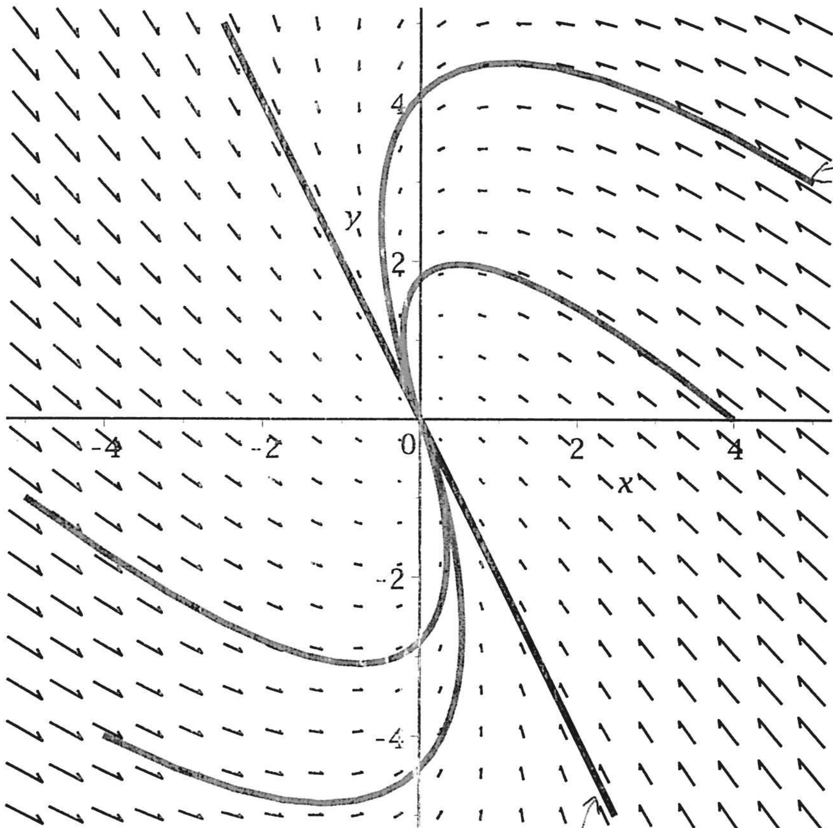
Solution set for v_2.

Can use any ~~sol~~ solution for v_2.
Let $v_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

Thus, our general solution is:

$$x = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t} + c_2 \left(\begin{pmatrix} 1 \\ -2 \end{pmatrix} t e^{-3t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{-3t} \right)$$

```
> with(DEtools):with(plots):
> solutioncurves:=phaseportrait([(D(x))(t) = -5*x(t)-y(t), (D(y))
  (t) = 4*x(t)-y(t)], [x(t), y(t)], t = 0 .. 2, [[x(0)=-5,y(0)=-1],
  [x(0)=5,y(0)=3],[x(0) = 4, y(0) = 0],[x(0)=-4,y(0)=-4],[x(0)=2.5,
  y(0)=-5],[x(0)=-2.5,y(0)=5]],linecolor=[red,red,red,red,black,
  black],color=black,arrows=none,x=-5..5,y=-5..5):
> vectorfield:=fieldplot([-5*x-y,4*x-y],x = -5 .. 5, y = -5 .. 5):
> display(cs,f);
```



a solution curve.

eigenspace for $r=-3$.

(0,0) is an improper node.

```
>
```

Notes ① The vector v_2 is called a generalized eigenvectors.
Notice

$$(A - rI)v_2 = v_1$$

$$\Rightarrow \quad (A-rI)(A-rI)v_2 = (A-rI)v_1 = 0$$

$$\Rightarrow \quad (A-rI)^2 v_2 = 0.$$

② The node at $(0,0)$ is called an improper node.

③ Exercise 18 (7.8) shows what to do if you get an eigenvalue of multiplicity 3 and a one-dimensional eigenspace.

④ Exercise 19 (7.8) shows what to do if you get an eigenvalue of mult. 3 with a two-dimensional eigenspace.