7.9 Nonhomogeneous Linear Systems.

<u>Method:</u> Decoupling through diagonalization

Suppose we are given $v' = Av + \underbrace{\begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}}_{g=}$, and $(*)$

that A is diagonalizable. Suppose the eigenvalues are $\lambda_1, \dots \lambda_n$ and $T = [$matrix of L.I. eigenvectors$]$.
Then we know

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = T^{-1}AT.$$

Let $u = T^{-1}v$ (change of variables).

Then $v = Tu$ and $v' = Tu'$. Since $A = TDT^{-1}$
$(*)$ becomes

$$Tu' = (TDT^{-1})(Tu) + g$$

these cancel

Multiply both sides by T^{-1}, to get

$$u' = Du + T^{-1}g$$

Let $h = T^{-1}g$. Then $u' = Du + h$ is a decoupled system:

$$u_1' = \lambda_1 u_1 + h_1(t)$$
$$\vdots$$
$$u_n' = \lambda_n u_n + h_n(t)$$

Each can be solved as a linear first order eq as in Ch 2. Once this is done convert to the original variables, i.e, $v = Tu$.

Example 1 $\quad v' = \begin{bmatrix} 4 & 3 \\ -6 & -5 \end{bmatrix} v + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Find general solution.

Find eigenvalues. $\begin{vmatrix} 4-\lambda & 3 \\ -6 & -5-\lambda \end{vmatrix} = (\lambda-4)(\lambda+5)+18$

$= \lambda^2+\lambda-20+18$

$= \lambda^2+\lambda-2 = (\lambda+2)(\lambda-1)$

$\Rightarrow \lambda = 1, -2.$

Find eigenvectors.

$\boxed{\lambda=1}$ $\quad \begin{bmatrix} 3 & 3 \\ -6 & -6 \end{bmatrix}$ $\quad 3a+3b=0 \quad a=-b \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ eigenvector.

$\boxed{\lambda=-2}$ $\quad \begin{bmatrix} 6 & 3 \\ -6 & -3 \end{bmatrix}$ $\quad 6a+3b=0$ $\quad \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ eigenvector.

$2a=-b$

Diagonalize the system.

Let $T = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$. Then $T^{-1} = \begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix}$.

Check that $D = T^{-1}AT = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$.

Let $h = T^{-1}g = \begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ -5 \end{bmatrix}$.

New system is decoupled $\quad \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} u_1 \\ -2u_2 \end{bmatrix} + \begin{bmatrix} -7 \\ -5 \end{bmatrix}$.

Solve the Decoupled System

$$u_1' = u_1 - 7$$

$$\frac{du}{dt} = u - 7$$

$$\int \frac{du}{u-7} = \int dt$$

$$\ln|u-7| = t + C$$

$$|u-7| = e^{t+C} = e^C e^t$$

$$u - 7 = \pm e^C e^t$$

$$u - 7 = C e^t$$

$$\underline{u_1 = C_1 e^t + 7}$$

$$u_2' = -2u_2 - 5$$

$$\frac{du}{dt} = -2u - 5$$

$$\int \frac{-2\,du}{-2u-5} = \int -2\,dt$$

$$\ln|-2u-5| = -2t + C$$

$$-2u - 5 = Ce^{-2t}$$

$$-2u = Ce^{-2t} + 5$$

$$\underline{u_2 = C_2 e^{-2t} - 5/2}$$

Recover original ~~natural~~ variables.

$$V = Tu = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -u_1 + u_2 \\ u_1 - 2u_2 \end{bmatrix}$$

$$= \begin{bmatrix} -C_1 e^t - 7 + C_2 e^{-2t} - 5/2 \\ C_1 e^t + 7 - 2C_2 e^{-2t} + 5 \end{bmatrix}$$

$$= C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^t + C_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} + \begin{bmatrix} -\frac{19}{2} \\ 12 \end{bmatrix}$$

Done!

Example 2 $v' = \begin{bmatrix} 2 & 2 \\ -6 & -5 \end{bmatrix} v + \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$. Find general solution.

Solution Find eigenvalues $\begin{vmatrix} 2-\lambda & 2 \\ -6 & -5-\lambda \end{vmatrix} = (\lambda-2)(\lambda+5)+12$

$= \lambda^2+3\lambda+2$

Find eigenvectors.

$= (\lambda+1)(\lambda+2)$

$\lambda = -1, -2.$

$\lambda = -1$ $\begin{bmatrix} 3 & 2 \\ -6 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$3a+2b=0.$ Use $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

$\lambda = -2$ $\begin{bmatrix} 4 & 2 \\ -6 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$4a+2b=0$

$2a+b=0$ Use $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Find transformation matrix T and T^{-1}. Diagonalize.

$T = \begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix}$. $T^{-1} = \begin{bmatrix} -2 & -1 \\ -3 & -2 \end{bmatrix}$

Check: $T^{-1}AT = \begin{bmatrix} -2 & -1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ ✓

Decouple)

$h = T^{-1} \begin{bmatrix} s \\ c \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} s \\ c \end{bmatrix} = \begin{bmatrix} -2s-c \\ -3s-2c \end{bmatrix}$

$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} -u_1 \\ -2u_2 \end{bmatrix} + \begin{bmatrix} -2s-c \\ -3s-2c \end{bmatrix}$

$u_1' = -u_1 - 2s - c$

$u_2' = -2u_2 - 3s - 2c$

Solve the decoupled system.

$u_1' = -u_1 - 2s - c$

$u' + u = -2s - c.$

Let $u_h = Ce^{-t}$. Let $u_p = As + Bc.$

Then $u_p' = Ac - Bs.$

$u_p' + u_p = (A-B)s + (A+B)c$

must $= -2s - c.$

Thus,
$$A - B = -2$$
$$A + B = -1$$
$$2A = -3$$
$$A = -3/2$$
$$B = -1 - A = \tfrac{1}{2}$$

Thus $u_1 = u_h + u_p = C_1 e^{-t} - \frac{3}{2}\sin t + \frac{1}{2}\cos t$

Next $u_2' = -2u_2 - 3s - 2c.$

$u' + 2u = -3s - 2c$

Let $u_h = Ce^{-2t}$. Let $u_p = As + Bc.$

Then $u_p' = Ac - Bs$

$u_p' + 2u_p = (2A - B)s + (2B + A)c$

must $= -3s - 2c.$

$$2A - B = -3$$
$$-2(A + 2B = -2)$$
$$-5B = 1$$
$$B = -1/5$$
$$A = -2 - 2B = -2 + \tfrac{2}{5} = \tfrac{-8}{5}$$

$u_p = \frac{-8}{5}s + \frac{-1}{5}c$

$u_2 = u_h + u_p = C_2 e^{-2t} - \frac{8}{5}\sin t - \frac{1}{5}\cos t.$

Finally, convert back to original variables.

$$V = Tu.$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} c_1 e^{-t} - \frac{3}{2}\sin t + \frac{1}{2}\cos t \\ c_2 e^{-2t} - \frac{8}{5}\sin t - \frac{1}{5}\cos t \end{bmatrix}$$

$$= \begin{bmatrix} -2c_1 e^{-t} + 3s - c + c_2 e^{-2t} - \frac{8}{5}s - \frac{1}{5}c \\ 3c_1 e^{-t} - \frac{9}{2}s + \frac{3}{2}c - 2c_2 e^{-2t} + \frac{16}{5}s + \frac{2}{5}c \end{bmatrix}$$

$$= c_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} + \begin{bmatrix} \frac{7}{5}\sin t - \frac{6}{5}\cos t \\ -\frac{13}{10}\sin t + \frac{19}{10}\cos t \end{bmatrix}$$

$\boxed{\text{Done!}}$

Check computer (Maple)

```
> dsolve({diff(x(t),t)=2*x(t)+2*y(t)+sin(t), diff(y(t),t)=-6*x(t)
  -5*y(t)+cos(t)}, [x(t),y(t)]);
```

$$\left\{x(t) = \frac{7}{5}\sin(t) - \frac{6}{5}\cos(t) + \frac{1}{2}e^{-2t}\_C1 - \frac{2}{3}e^{-t}\_C2,\ y(t) = \frac{19}{10}\cos(t) - \frac{13}{10}\sin(t) - e^{-2t}\_C1 + e^{-t}\_C2\right\}$$

Looks different, but it is the same. Can you see that?

Example 3 with complex eigenvalues.

Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $g(t) = \begin{bmatrix} 2\sin\omega t \\ 2e^{2t} \end{bmatrix}$.

Find the general solution to $v' = Av + g$.

Solution Show $\lambda = 1 \pm i$ are the eigenvalues.

Show $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$ are corresponding eigenvectors.

Decouple Let $T = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$. Then $T^{-1} = \frac{1}{2i}\begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$.

Check $T^{-1}AT = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}$.

Let $u = T^{-1}v$ be new variables. Then $v = Tu$ and $v' = Tu'$.
Hence

$$Tu' = ATu + g$$

$$\Rightarrow \quad u' = T^{-1}ATu + T^{-1}g$$

$$\Rightarrow \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \sin t + ie^{2t} \\ \sin\omega t - ie^{2t} \end{bmatrix}$$

$$\Rightarrow \quad ① \quad u_1' = (1+i)u_1 + \sin\omega t + ie^{2t}$$

$$② \quad u_2' = (1-i)u_2 + \sin\omega t - ie^{2t}$$

Solve ① for $u_1(t)$.

$$u_1' - (1+i)u_1 = \sin t + ie^{2t}.$$

$u_h = C_1 e^{(1+i)t}$. Let $u_p = A\sin t + B\cos t + De^{2t}$.

Then $u_p' = Ac - Bs + 2De^{2t}$

$$u_p' - (1+i)u_p = (-B-(1+i)A)s + (A-(1+i)B)c + (2D-(1+i)D)e^{2t}$$

must $=$ $s + ie^{2t}$.

Thus, $-B-(1+i)A = 1$

and $A-(1+i)B = 0 \Rightarrow A = (1+i)B$.

$$\Rightarrow -B-(1+i)(1+i)B = 1$$

$$-B(1+(1+i)^2) = 1$$

$$B(1+1+2i-1) = -1$$

$$B = \frac{-1}{1+2i}\frac{(1-2i)}{(1-2i)} = \frac{-1+2i}{5}$$

$$A = (1+i)B = \frac{(1+i)(-1+2i)}{5} = \frac{-3+i}{5}$$

Next, $2D-(1+i)D = i$

$$D(2-1-i) = i$$

$$D = \frac{i}{1-i}\frac{(1+i)}{(1+i)} = \frac{-1+i}{2}$$

Thus, $u_1(t) = C_1 e^{(1+i)t} + \left(\frac{-3+i}{5}\right)\sin t + \left(\frac{-1+2i}{5}\right)\cos t + \left(\frac{-1+i}{2}\right)e^{2t}$

Solve ② for $u_2(t)$, but we know that answer.

$$u_2(t) = C_2 e^{(1-i)t} + \left(\frac{-3-i}{5}\right)\sin t + \left(\frac{-1-2i}{5}\right)\cos t + \left(\frac{-1-i}{2}\right)e^{2t}.$$

Convert back to original variables

$$V = Tu = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} C_1 e^{(1+i)t} + \left(\frac{-3+i}{5}\right) s + \left(\frac{-1+2i}{5}\right) c + \left(\frac{-1+i}{2}\right) e^{2t} \\ C_2 e^{(1-i)t} + \left(\frac{-3-i}{5}\right) s + \left(\frac{-1-2i}{5}\right) c + \left(\frac{-1-i}{2}\right) e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} C_1 e^{(1+i)t} + C_2 e^{(1-i)t} - \frac{6}{5} s - \frac{2}{5} c - e^{2t} \\ -iC_1 e^{(1+i)t} + iC_2 e^{(1-i)t} + \frac{2}{5} s + \frac{4}{5} c + e^{2t} \end{bmatrix}$$

Rewrite as real valued functions.

V_1 : $\quad C_1 e^{(1+i)t} + C_2 e^{(1-i)t} = (C_1+C_2)\cos t\, e^t + i(C_1-C_2)\sin t\, e^t$

V_2 : $\quad -iC_1 e^{(1+i)t} + C_2 e^{(1-i)t} = -i(C_1-C_2)\cos t\, e^t + (C_1+C_2)\sin t\, e^t$.

Let $P_1 = C_1 + C_2$ and $P_2 = i(C_1 - C_2)$.

Now

$$V_1 = P_1 \cos t\, e^t + P_2 \sin t\, e^t - \frac{6}{5}\sin t - \frac{2}{5}\cos t - e^{2t}$$

$$V_2 = -P_2 \cos t\, e^t + P_1 \sin t\, e^t + \frac{2}{5}\sin t + \frac{4}{5}\cos t + e^{2t}$$

Check with computer (Maple)

```
> dsolve({diff(x(t),t)=x(t)-y(t)+2*sin(t), diff(y(t),t)=x(t)+y(t)
  +2*exp(2*t)}, [x(t),y(t)]);
```

$$\left\{x(t) = e^t \cos(t)\,\_C2 - e^t \sin(t)\,\_C1 - \frac{6}{5}\sin(t) - \frac{2}{5}\cos(t) - e^{2t},\ y(t) = e^t \sin(t)\,\_C2 + e^t \cos(t)\,\_C1 + \frac{4}{5}\cos(t) + \frac{2}{5}\sin(t) + e^{2t}\right\}$$

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Example (4) $V' = \begin{bmatrix} 8 & 4 \\ -9 & -4 \end{bmatrix} V + \begin{bmatrix} \sin t \\ t \end{bmatrix}$. Find general solution.
↳ with repeated eigenvalues.

Solution Find eigenvalues. $\begin{vmatrix} 8-\lambda & 4 \\ -9 & -4-\lambda \end{vmatrix} = (\lambda-8)(\lambda+4)+36$

$= \lambda^2 - 4\lambda - 32 + 36$

$= \lambda^2 - 4\lambda + 4 = (\lambda-2)^2.$

$\lambda = 2, 2.$ Yikes!

Find eigenvector(s).

If there are two L.I. eigenvectors, we know what to do.

If not ....

Let $\lambda = 2$. $\begin{bmatrix} 6 & 4 \\ -9 & -6 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. $6a + 4b = 0$

$3a + 2b = 0$

Use $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

No other L.I e.vectors!

What to do? Find generalize eigenvector.

$$\begin{bmatrix} 6 & 4 \\ -9 & -6 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$6a + 4b = -2$

$-9a - 6b = 3$ these are compatible. and redundant.

$3a + 2b = -1$

$a = -\frac{2}{3}b - \frac{1}{3}$

$b = b$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1 \end{bmatrix} b + \begin{bmatrix} -1/3 \\ 0 \end{bmatrix}$$

Use any value for b. I let $b = 0$.

Now $\begin{bmatrix} -1/3 \\ 0 \end{bmatrix}$ is my generalized e.v.

Put $\begin{bmatrix} 8 & 4 \\ -9 & -4 \end{bmatrix}$ into Jordan form.

Let $T = [\text{eigenvector} \;\; \text{generalized e.v.}] = \begin{bmatrix} -2 & -1/3 \\ 3 & 0 \end{bmatrix}$.

Then $T^{-1} = \begin{bmatrix} 0 & 1/3 \\ -3 & -2 \end{bmatrix}$.

Check: $T^{-1}AT = \begin{bmatrix} 0 & \frac{1}{3} \\ -3 & -2 \end{bmatrix} \begin{bmatrix} 8 & 4 \\ -9 & -4 \end{bmatrix} \begin{bmatrix} -2 & -\frac{1}{3} \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = J$.

Next let $h = T^{-1}g$ as before.

$$h = \begin{pmatrix} 0 & \frac{1}{3} \\ -3 & -2 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{bmatrix} t/3 \\ -3s-2t \end{bmatrix}$$

Now we have, letting $u = T^{-1}v$, that $u' = Ju + h$.

$$u_1' = 2u_1 + u_2 + t/3$$

$$u_2' = 2u_2 - 3s - 2t.$$

It is not completely decoupled. But u_2' is not dependent on u_1. So, we can solve for u_2. Then we plug this into the u_1' equation, and solve it.

Solve for u_2

$u' - 2u = -3s - 2t.$

$u_h = Ce^{2t}$. Let $u_p = As + Bc + Dt + E.$

Then $u_p' = Ac - Bs + D.$

Now $u_p' - 2u_p = (-2A - B)s + (A - 2B)c - 2Dt + D - 2E.$

which must $= \quad -3s \quad\quad -2t.$

$-2A - B = -3$ $\quad$ $-2D = -2$ $\quad$ $D - 2E = 0$

$A - 2B = 0$ $\quad$ $D = 1$ $\quad$ $E = \frac{1}{2}$

$-5B = -3$

$B = 3/5$

$A = 2B = 6/5.$

Thus $u_2 = C_2 e^{2t} + \frac{6}{5}\sin t + \frac{3}{5}\cos t + t + \frac{1}{2}.$

Solve for u_1

$u_1' = 2u_1 + u_2 + t/3$

$u_1' - 2u_1 = C_2 e^{2t} + \frac{6}{5}\sin t + \frac{3}{5}\cos t + \frac{4}{3}t + \frac{1}{2}$

Let $u_h = C_1 e^{2t}$. Let $u_p = Pte^{2t} + As + Bc + Dt + E$

Plug u_p in: $u_p' = Pe^{2t} + 2Pte^{2t} + Ac - Bs + D$

$u_p' - 2u_p = Pe^{2t} + 0te^{2t} + (-2A - B)s + (A - 2B)c - 2Dt + D - 2E$

must $= C_2 e^{2t} + \frac{6}{5}s + \frac{3}{5}c + \frac{4}{3}t + \frac{1}{2}$

Thus, $P = C_2$.

$-2A - B = \frac{6}{5}$
$A - 2B = 3/5$

$-5B = \frac{12}{5}$

$B = \frac{-12}{25}$

$A = \frac{3}{5} + 2B = \frac{15}{25} + \frac{-24}{25} = \frac{-9}{25}$

$-2D = \frac{4}{3} \Rightarrow D = -\frac{2}{3}$

$D - 2E = \frac{1}{2}$

$-\frac{2}{3} - 2E = \frac{1}{2}$

$-2E = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$
$\frac{3}{6} \quad \frac{4}{6}$

$E = \frac{-7}{12}$

Thus, $u_1 = C_1 e^{2t} + C_2 t e^{2t} - \frac{9}{25} \sin t + \frac{-12}{25} \cos t - \frac{2}{3} t - \frac{7}{12}$.

Finally, convert back to the original variables $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

Recall $v = T u$. Thus,

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2 & -\frac{1}{3} \\ 3 & 0 \end{bmatrix} \begin{bmatrix} c_1 e^{2t} + c_2 t e^{2t} - \frac{9}{25} s - \frac{12}{25} c - \frac{2}{3} t - \frac{7}{12} \\ c_2 e^{4} + \frac{6}{5} s + \frac{3}{5} c + t + \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \left(-2c_1 - \frac{c_2}{3}\right) e^{2t} - 2c_2 t e^{2t} + \left(\frac{18}{25} - \frac{2}{5}\right) s + \left(\frac{24}{25} + \frac{1}{5}\right) c + \left(\frac{4}{3} - \frac{1}{3}\right) t + \frac{14}{12} - \frac{1}{6} \\ 3c_1 e^{2t} + 3c_2 t e^{2t} - \frac{27}{25} s - \frac{36}{25} c - 2t - \frac{7}{4} \end{bmatrix}$$

$$= \begin{bmatrix} -2c_1 - \frac{c_2}{3} \\ 3c_1 \end{bmatrix} e^{2t} + \begin{bmatrix} -2c_2 \\ 3c_2 \end{bmatrix} t e^{2t} + \begin{bmatrix} \frac{8}{25} \\ -\frac{27}{25} \end{bmatrix} \sin t + \begin{bmatrix} \frac{29}{25} \\ -\frac{36}{25} \end{bmatrix} \cos t + \begin{bmatrix} 1 \\ -2 \end{bmatrix} t + \begin{bmatrix} 1 \\ -\frac{7}{4} \end{bmatrix}$$

This is the general solution.

Done!

Compare with computer (Maple)

```
> dsolve({diff(x(t),t)=8*x(t)+4*y(t)+sin(t), diff(y(t),t)=-9*x(t)
  -4*y(t)+t}, [x(t),y(t)]);
```

$$\left\{x(t) = -\frac{2}{3} e^{2t}\_C2 - \frac{2}{3} e^{2t} t\_C1 - \frac{1}{9} e^{2t}\_C1 + \frac{8}{25} \sin(t) + \frac{19}{25} \cos(t) + 1 + t,\right.$$
$$\left. y(t) = e^{2t}\_C2 + e^{2t} t\_C1 - \frac{36}{25} \cos(t) - \frac{27}{25} \sin(t) - 2t - \frac{7}{4}\right\}$$

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I made one mistake. Find it!