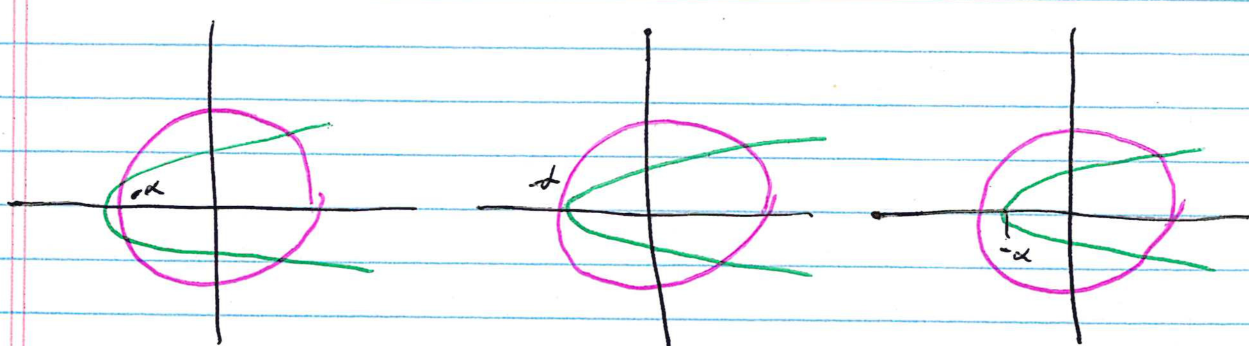
A Bifurcation Example

More on Example 2. Consider $x' = x^2+y^2-9$,
$y' = 3y^2+\alpha - x$.

When $\alpha=-3$ we have Example 2 and $(0,-3)$ is a critical pt. Recall ~~we~~ the linearization had eigenvalues -6 and 0. But numerically the vector field was only $\begin{bmatrix}\delta\\0\end{bmatrix}$ at $(0,-3)$. Just above this pt solution move up away, and just below that move up toward $(0,-3)$.

Now if $\alpha=-2.9$ there would be no cr. pt at or near $(0,-3)$; there would only to two cr. pts. If $\alpha=-3.1$ there would be four cr. pts. See figure

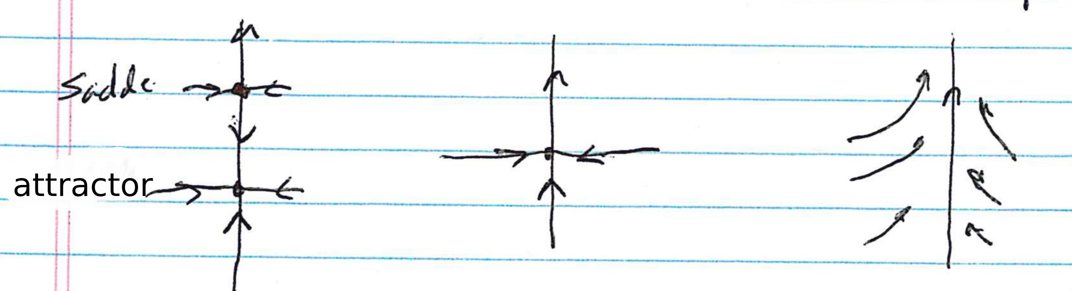


$\alpha=+3.1$ $\qquad$ $\alpha=+3$ $\qquad$ $\alpha=+2.9$

Green parabola $\Leftrightarrow y'=0$. Red circle $\Leftrightarrow x'=0$.

This is an example of a <u>bifurcation</u>.

You can check that the two "new" cr. pts are a saddle & attactor.



$x=-\alpha=-3.1$ $\qquad$ $y=-\alpha=-3$ $\qquad$ $x=-\alpha=-2.9$

In fact, this system will have several other bifurcations as we very α.

$y'=0 \iff x=3y^2+\alpha \iff y^2=\frac{x-\alpha}{3}$.

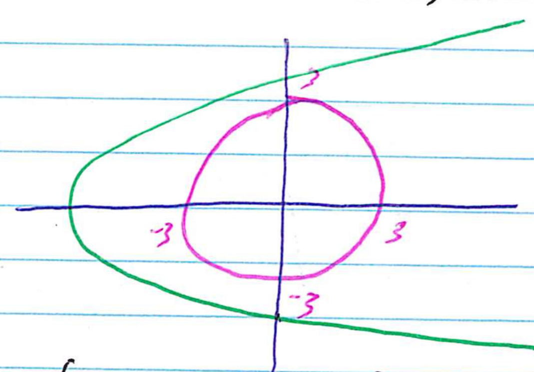
so, $x^2+\left(\frac{x-\alpha}{3}\right)-9=0$ when $x'=0$ and $y'=0$.

Thus, $3x^2+x-\alpha-27=0$

or $x=\frac{-1\pm\sqrt{1+12(\alpha+27)}}{6}$.

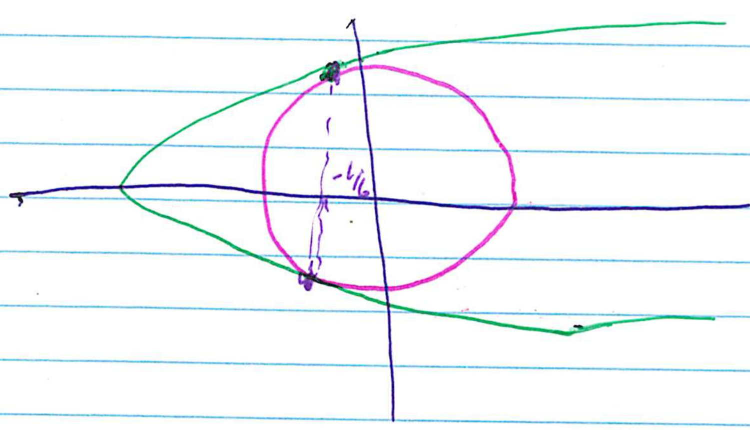
1. If $1+12(\alpha+27)<0$ $\left(\alpha<-\frac{1}{12}-27=\frac{-323}{12}=-26.91\bar{6}.\right)$

then there are no real solutions for x, hence there are no cr. pt.



2. If $1+12(\alpha+27)=0$ $(\alpha=-26.91\bar{6})$, there is one real solution for x, $x=-\frac{1}{6}$. Then $y^2=\frac{-\frac{1}{6}+26.91\bar{6}}{3}$

$\Rightarrow y\approx\pm 2.9861$



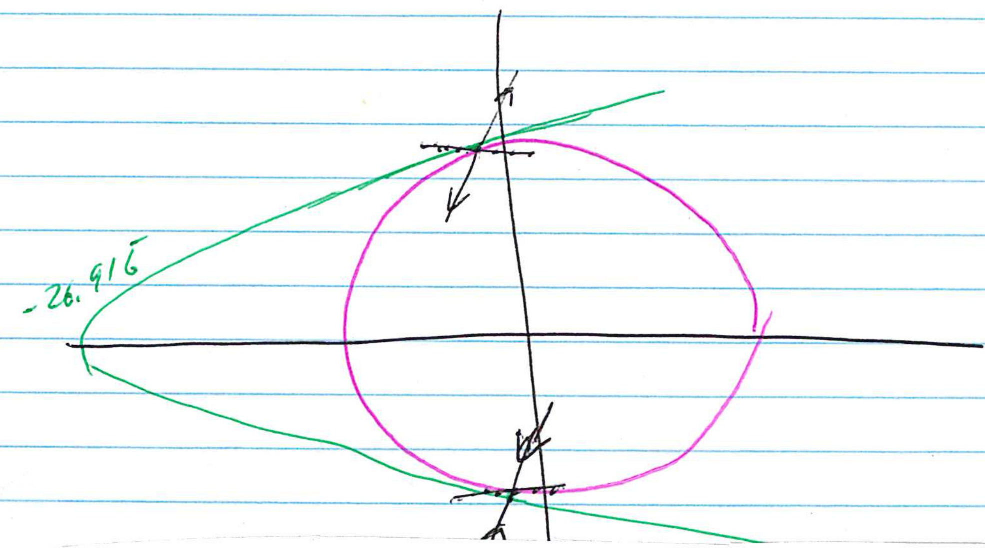
I computed the linearizations. Each has a zero eigenvalue. This is typical a bifurcations.

For $x = -\frac{1}{6}$, $y = +\sqrt{\frac{107}{12}}$ e.values are 0, $-\frac{1}{3}+\sqrt{321}$.

Corresponding e.vectors are $\begin{bmatrix} \sqrt{321} \\ 1 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$.

For $x = -\frac{1}{6}$, $y = -\sqrt{\frac{107}{12}}$ I got 0 $\begin{bmatrix} -\sqrt{321} \\ 1 \end{bmatrix}$, $-\frac{1}{3}-\sqrt{321}$ $\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$

$$\cancel{0, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } -\frac{1}{3}-\sqrt{321}, \begin{bmatrix} -\sqrt{321} \\ 1 \end{bmatrix}.}$$




```
> with(LinearAlgebra):
> Eigenvectors(Matrix([[-1/3, 2*sqrt(107/12)],[-1, sqrt(321)]]));
```

$$\begin{bmatrix} 0 \\ -\frac{1}{3}+\sqrt{321} \end{bmatrix}, \begin{bmatrix} \sqrt{321} & \frac{1}{3} \\ 1 & 1 \end{bmatrix}$$

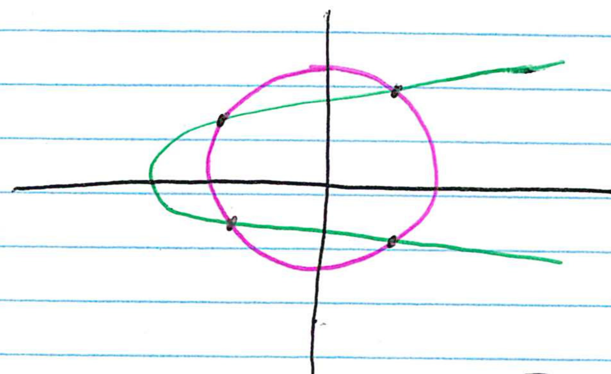
```
> Eigenvectors(Matrix([[-1/3, -2*sqrt(107/12)],[-1, -sqrt(321)]]));
```

$$\begin{bmatrix} -\frac{1}{3}-\sqrt{321} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & -\sqrt{321} \\ 1 & 1 \end{bmatrix}$$

3. If $1 + 12(\alpha+27) > 0$ $(\alpha > -26.91\bar{6})$

and $\frac{x-\alpha}{3} > 0$, there are four cr. pts. See graph.

This is another bifurcation at $\alpha = -26.91\bar{6}$.



Four critical points.

Assume $-3 \cancel{\geq} \alpha > -26.91\bar{6}$.

Then $x = \frac{-1 \pm \sqrt{1+12(\alpha+27)}}{6}$.

Let's look at $(-)$ case. Let $g(\alpha) = y^2 = \frac{x-\alpha}{3} = \frac{-1-\sqrt{1+12(\alpha+27)} - \alpha}{3}$

At $\alpha=-3$, $g(-3) = 0$. Notice $g(\alpha)$ decreases as α increases.

Thus $g(\alpha) \cancel{\geq} 0$ for $\alpha < -3$. Hence there are

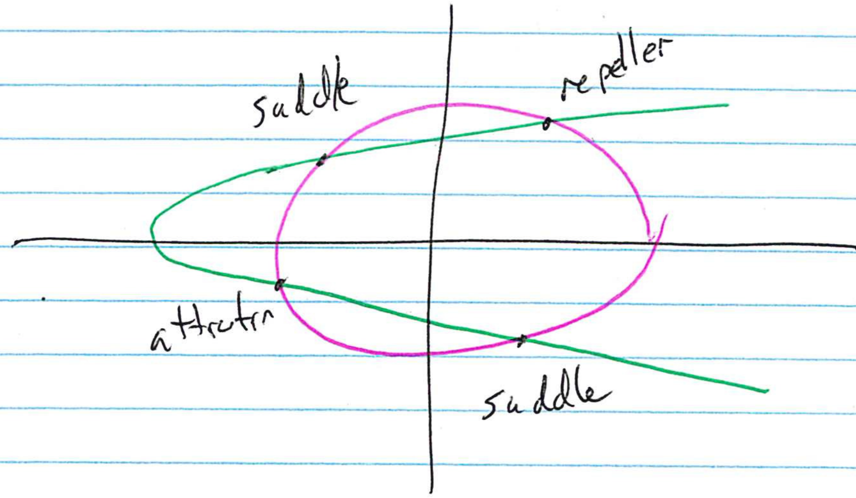
two real values for y in the $(-)$ case.

For $x = \frac{-1+\sqrt{}}{6}$, x is larger so $\frac{x-\alpha}{3}$ is

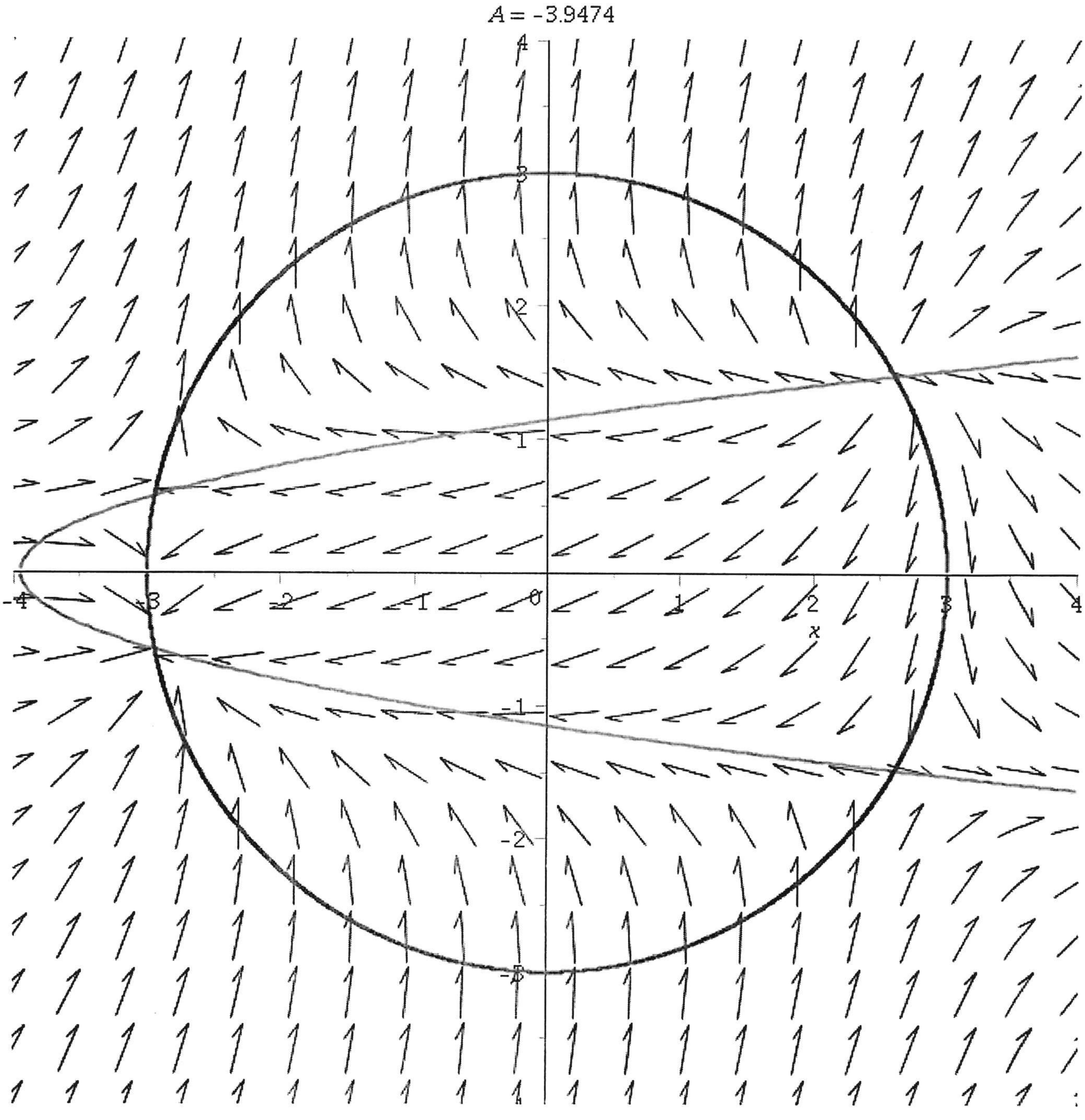
still pos. Hence, we have two more cr. pts,

for a total of four as in the graph.

I studied the vector field plots and observed that

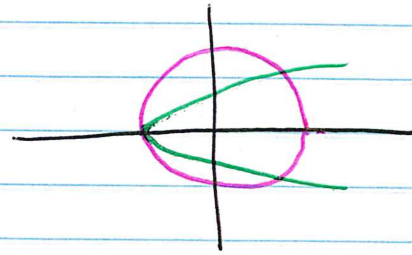


See next page. You should be able to verify the stability types of the four cr. pts.



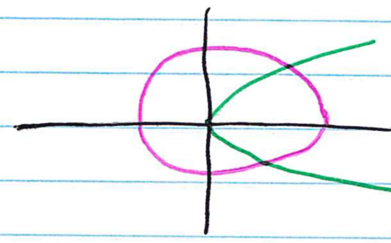
A = -3.9474
4
3
2
1
-1
-2
-4
-3
-2
-1
0
1
2
3
4
x

4. If $\alpha = -3$, we have the three cr. pts we found in Example 2.



5 If $3 > \alpha > -3$, the graph shows just two cr. pts.

We can check this.



$$x = \frac{-1 \pm \sqrt{1+12(\alpha+27)}}{6}$$

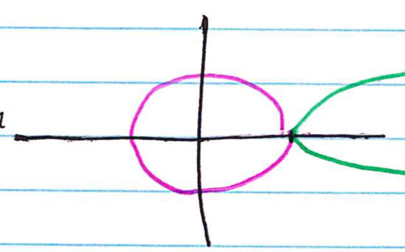
The $(-)$ case will have $\frac{x-\alpha}{3} < 0$, so no real solutions for $y^2 = \frac{x-\alpha}{3}$. For the $(+)$ case consider $\alpha = 3$.

Then $x = \frac{-1+19}{6} = 3$. Thus $(3,0)$ would be cr. pt.

For $3 > \alpha > -3$, $\frac{x_+ - \alpha}{3} > 0$ so there are two solutions for $y^2 = \frac{x_+ - \alpha}{3}$.

6. $\alpha = 3$, we just looked at this. This is another bifurcation value.

Check that the linearization has a 0 eivalue.



7. $\alpha > 3$. No cr. pts.