Section 9.3
Linearization

First, we provide some background material that is not in the textbook.

Recall, for a vector $\mathbf{v} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, that

$$||\mathbf{v}|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Definition. Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^m$. We may write

$$f(x_1, x_2 \ldots, x_n) = (f_1(x_1, x_2 \ldots, x_n), f_2(x_1, x_2 \ldots, x_n), \ldots, f_m(x_1, x_2 \ldots, x_n)).$$

We say that f is **differentiable** at $\mathbf{v}_o \in U$ if there exists an $m \times n$ matrix T such that

$$\lim_{\mathbf{v} \to \mathbf{v}_o} \frac{||f(\mathbf{v}) - f(\mathbf{v}_o) - T(\mathbf{v} - \mathbf{v}_o)||}{||\mathbf{v} - \mathbf{v}_o||} = 0. \qquad (*)$$

We call T the derivative of f at $\mathbf{v}_o$ and use the notation $T = Df(v_o)$. Note: we are regarding $\mathbf{v}$ and $\mathbf{v}_o$ has column vectors now.

In the one variable case, $f : \mathbb{R} \to \mathbb{R}$, this is equivalent to

$$\lim_{x \to x_o} \frac{|f(x) - f(x_o) - m(x - x_o)|}{|x - x_o|} = 0,$$

which is equivalent to

$$\lim_{x \to x_o} \frac{f(x) - f(x_o)}{x - x_o} = m = f'(x_o).$$

Theorem. Suppose, $U \subset \mathbb{R}^n$ is open and $f : U \to \mathbb{R}^m$ is differentiable at $\mathbf{v}_o \in \mathbb{R}^n$. Then all partial derivatives of the f_i, the components of f, exist at $\mathbf{v}_o$ and

$$Df(v_o) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix},$$

with all entries evaluated at $\mathbf{v}_o$. This matrix is often called the **Jacobian** of f.

Outline of Proof. Let $\mathbf{h} = \mathbf{v} - \mathbf{v}_o$. Then $(*)$ becomes

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{||f(\mathbf{v}_o + \mathbf{h}) - f(\mathbf{v}_o) - T\mathbf{h}||}{||\mathbf{h}||} = 0.$$

It can be shown that this is equivalent to

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{|f_i(\mathbf{v}_o + \mathbf{h}) - f_i(\mathbf{v}_o) - (T\mathbf{h})_i|}{||\mathbf{h}||} = 0,$$

for $i = 1, \ldots, m$, where $(T\mathbf{h})_i$ is the i-th entry of the column vector $T\mathbf{h}$.

We are given that these m limits exist. So, we can compute the limits along any path where $\mathbf{h} \to \mathbf{0}$ and the result is the same. Let $\mathbf{h} = a\mathbf{e}_j$. (Recall $\mathbf{e}_j = \begin{bmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{bmatrix}^T$, where the 1 is in the j-th position.) Then we have

$$\lim_{a\to 0} \frac{|f_i(\mathbf{v}_o + a\mathbf{e}_j) - f_i(\mathbf{v}_o) - a(T\mathbf{e}_j)_i|}{|a|} = 0,$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. This is equivalent to

$$\lim_{a\to 0} \frac{f_i(\mathbf{v}_o + a\mathbf{e}_j) - f_i(\mathbf{v}_o)}{a} = (T\mathbf{e}_j)_i.$$

Now, $T\mathbf{e}_j$ is just the j-th column of T, and $(T\mathbf{e}_j)_i$ is its i-th entry. Thus, $(T\mathbf{e}_j)_i = T_{ij}$. The LHS is just the definition of the partial derivative $\partial f_i/\partial x_j$. Thus,

$$T_{ij} = \frac{\partial f_i}{\partial x_j},$$

as claimed. □

Theorem. Let $f : U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^n$ is open. Suppose, the partial derivatives $\partial f_i/\partial x_j$ exist and are continuous in some open ball around $\mathbf{v}_o$. Then f is differentiable at $\mathbf{v}_o$.

Proof. See *Vector Calculus*, by Marsden and Tromba, 3-rd Ed., Section 2.7, Theorem 9.

Example. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = 3xyz^2$. Then

$$Df = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix} = \begin{bmatrix} 3yz^2 & 3xz^2 & 6xyz \end{bmatrix}.$$

You may notice that this is the **gradient** of f.

Example. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = \begin{bmatrix} x^2 + 2y \\ 3x + 2y \end{bmatrix}$. Find $Df(0, 0)$.

Solution. $Df(x, y) = \begin{bmatrix} 2x & 2 \\ 3 & 2 \end{bmatrix}$, which at $(0, 0)$ is $\begin{bmatrix} 0 & 2 \\ 3 & 2 \end{bmatrix}$.

Taylor's Theorem. Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$, and assume f is differentiable at $v_o \in U$. Then

$$f(\mathbf{v}_o + \mathbf{h}) = f(\mathbf{v}_o) + Df(\mathbf{v}_o) \cdot \mathbf{h} + R_1(\mathbf{v}_o, \mathbf{h}),$$

where

$$\lim_{\mathbf{h} \to 0} \frac{R_1(\mathbf{v}_o, \mathbf{h})}{||\mathbf{h}||} = 0.$$

Proof. See *V.C.*, page 243.

Now we return to the textbook.

Definition. Let $H(x, y) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$. Here f and g are real valued functions. We say H is **almost linear** or is **linearizable** at (x_o, y_o) if $H(x - o, y_o) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and

$$H(x, y) = A \begin{bmatrix} x - x_o \\ y - y_o \end{bmatrix} + \begin{bmatrix} r_f(x, y) \\ r_g(x, y) \end{bmatrix},$$

where A is a 2×2 constant matrix, for $R(x, y) = \begin{bmatrix} r_f(x, y) \\ r_g(x, y) \end{bmatrix}$ we have

$$\lim_{(x,y) \to (x_o, y_o)} \frac{||R(x, y)||}{||(x, y) - (x_o, y_o)||} = 0.$$

The matrix A is called the **linearization** of H at (x_o, y_o). This definition can easily be generalized to the $n \times n$ case.

Example. Let $H(x, y) = \begin{bmatrix} 2x - y + x^2 - xy^2 \\ x + 3y + xy + y^7 \end{bmatrix}$. Linearize H at $(0, 0)$.

Solution. Clearly, $H(0, 0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Notice $H(x, y) = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x^2 - xy^2 \\ xy + y^7 \end{bmatrix}$.

Let $R(x,y) = \begin{bmatrix} x^2 - xy^2 \\ xy + y^7 \end{bmatrix}$.

We need show $\frac{||R(x,y)||}{||(x,y)||} \to 0$ as $(x,y) \to (0,0)$. This is not hard to see using polar coordinates. Let $c = \cos\theta$ and $s = \sin\theta$.

$$\begin{aligned}
\frac{||R(x,y)||}{||(x,y)||} &= \left(\frac{(x^2-xy^2)^2 + (xy+y^7)^2}{x^2+y^2}\right)^{1/2} \\
&= \left(\frac{x^4 - 2x^3y^2 + x^2y^4 + x^2y^2 + 2xy^8 + y^{14}}{x^2+y^2}\right)^{1/2} \\
&= \left(\frac{r^4c^4 - 2r^5c^3s^2 + r^6c^2s^4 + r^4c^2s^2 + 2r^9cs^8 + r^{14}s^{14}}{r^2}\right)^{1/2} \\
&= \left(r^2c^4 - 2r^3c^3s^2 + r^4c^2s^4 + r^2c^2s^2 + 2r^7cs^8 + r^{12}s^{14}\right)^{1/2} \\
&\le r^2c^2 + r\sqrt{2r}|c|\sqrt{|c|}|s| + r^2|c|s^2 + r|cs| + r^3\sqrt{2r}\sqrt{|c|}s^4 + r^6|s^7| \\
&\le r^2 + r\sqrt{2r} + r^2 + r + r^3\sqrt{2r} + r^6 \\
&\to 0.
\end{aligned}$$

The first inequality is the **Triangle Inequality** (look it up if you need to). The second is because sine and cosine never exceed 1 in magnitude.

BTW, notice that $DH(0,0) = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$. Thus, we could have just invoked Taylor's Theorem. □

We want to apply these ideas to nonlinear differential equations. Given

$$\mathbf{v}' = H(\mathbf{v}(t))$$

The **critical points** (also called **rest points**, **fixed points**. or **equilibrium points**) are points in the phasespace where $\mathbf{v}' = \mathbf{0}$. So, we first find the critical points and then try to linearize at each of these.

Example 0. Find the critical points for the system below and find the linearization at each.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2x + y - 3 \\ x - 3y + 2 \end{bmatrix}.$$

Solution. It is easy to see that the only critical point is $(1,1)$.

The Jacobian matrix is $\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$ at all points, thus this is the linearization matrix at $(1,1)$. In fact, the original system is equivalent to

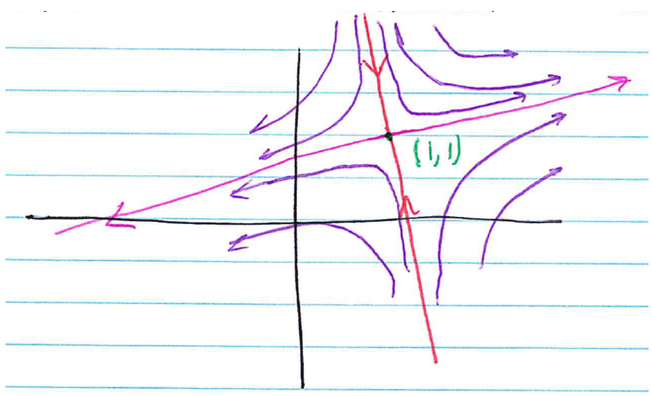
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \end{bmatrix}.$$

(Check this.)

The eigenvalues and eigenvectors of the Jacobian are

$$-\frac{1}{2} \pm \frac{\sqrt{29}}{2}, \quad \begin{bmatrix} 5 \pm \sqrt{29} \\ 2 \end{bmatrix}$$

Thus, near $(1,1)$ this system will behave like a saddle. See the sketch below. □



In fact, this system is a true saddle. It can be solved using the methods in Section 7.9 or simply made linear by a change of variables.

Example 1. Find the critical points for the system below and find the linearization at each. Sketch the system and compare with a computer.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2x + y^2 \\ x + y + xy \end{bmatrix}.$$

Solution. Step 1. Find the critical points.

$$\begin{aligned} 2x + y^2 = 0 \quad &\Longrightarrow \quad x = -\frac{y^2}{2}. \\ x + y + xy = 0 \quad &\Longrightarrow \quad -\frac{y^2}{2} + y - \frac{y^3}{2} = 0, \\ &\Longrightarrow \quad y^3 + y^2 - 2y = 0, \\ &\Longrightarrow \quad y(y+2)(y-1) = 0. \\ &\Longrightarrow \quad y = 0, y = -2 \text{ or } y = 1. \end{aligned}$$

Thus, the critical points are $(0,0)$, $(-2,-2)$ and $(-\frac{1}{2},1)$

Step 2. Lineraize at each critical point. The Jacobian is

$$J = \begin{bmatrix} 2 & 2y \\ 1+y & 1+x \end{bmatrix}.$$

Then we have

$$J(0,0) = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, J(-2,-2) = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \;\&\; J(-\tfrac{1}{2},1) = \begin{bmatrix} 2 & 2 \\ 2 & \frac{1}{2} \end{bmatrix}.$$

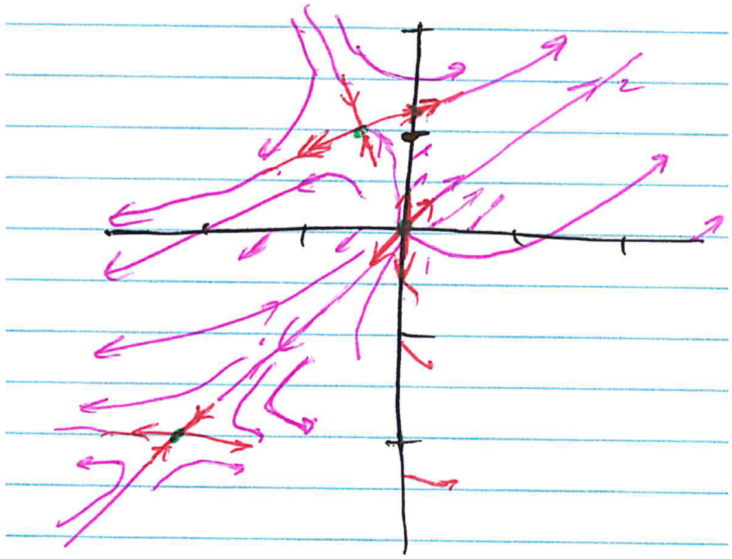
Step 3. Find the eigenvalues and eigenvectors for each of these.

$(0,0)$	$2, \begin{bmatrix} 1 \\ 1 \end{bmatrix};$	$1, \begin{bmatrix} 0 \\ 1 \end{bmatrix};$	unstable node, repller.
$(-2,-2)$	$-2, \begin{bmatrix} 1 \\ 1 \end{bmatrix};$	$3, \begin{bmatrix} -4 \\ 1 \end{bmatrix};$	saddle point, unstable.
$(-\frac{1}{2},1)$	$\frac{5+\sqrt{73}}{4}, \begin{bmatrix} 8 \\ \sqrt{73}-3 \end{bmatrix};$	$\frac{5-\sqrt{73}}{4}, \begin{bmatrix} 8 \\ -\sqrt{73}-3 \end{bmatrix};$	saddle point, unstable.

Step 4. Draw local patches at the critical points. Try to fit it all together.

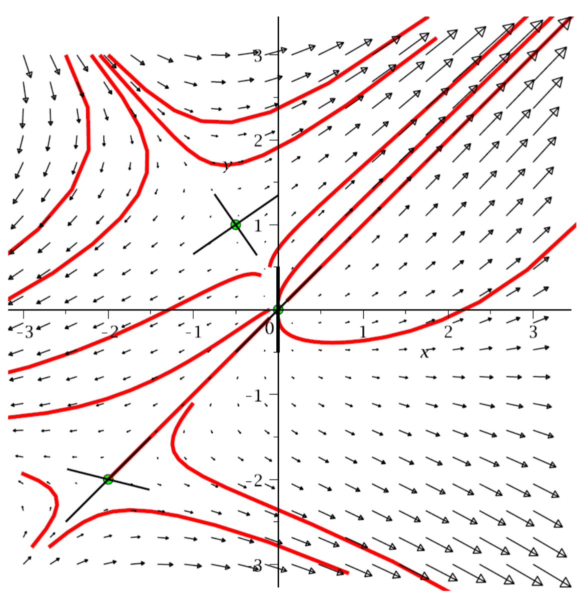
Additional Observations:

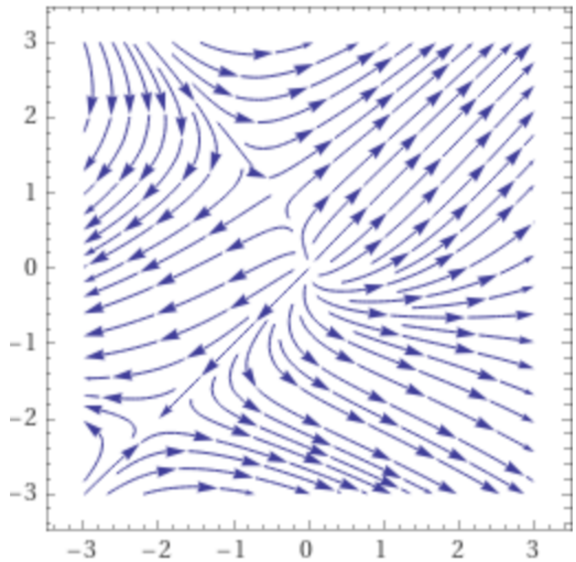
For $x=0$ and $y<0$ we can see that $x'>0$ and $y'=y<0$.
For $y=0$ and $x>0$ we can see that $x'=2x>0$ and $y'=x>0$.
For $x=0$ and $y>0$ we can see that $x'=y^2>0$ and $y'=y>0$.
For $y=0$ and $x<0$ we can see that $x'=2x<0$ and $y'=x<0$.
Think about $x=y$.



Step 5. Compare with computer.

The first plot was done with Maple. The code used is on the website for the course under the link for Section 9.3 as Example 1. The second was done on Wolfram Alpha using the widget at this link:
`https://www.wolframalpha.com/widgets/view.jsp?id=9298fea31cf266903b3df7174b95ddd7`





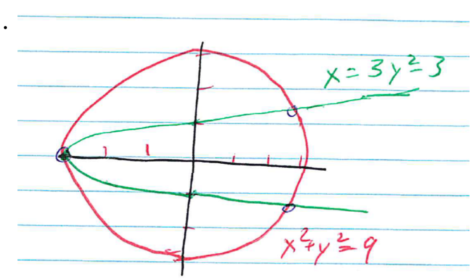
Example 2. Find the critical points for the system below and find the linearization at each.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x^2 + y^2 - 9 \\ 3(y^2 - 1) - x \end{bmatrix}.$$

Solution. Step 1. Find the critical points.

$x' = 0 \implies x^2 + y^2 = 9$

$y' = 0 \implies x = 3y^2 - 3$



See the graph. We should be able to find 3 critical points.

$y^2 = \frac{x+3}{3} \implies x^2 + \frac{x+3}{3} = 9 \implies 3x^2 + x - 24 = 0.$

Thus,

$$x = \frac{-1 \pm \sqrt{1 - 4 \cdot 3 \cdot (-24)}}{6} = \frac{-1 \pm \sqrt{289}}{6} = \frac{-1 \pm 17}{6} = \frac{8}{3} \text{ or } -3.$$

Now, if $x = -3$, then $y^2 = 0$ and so $y = 0$. Thus, $(-3, 0)$ is a critical point.

While if $x = \frac{8}{3}$, we have $y^2 = \frac{17}{9}$, so $y = \pm\frac{\sqrt{17}}{3}$. Thus, $(\frac{8}{3}, \pm\frac{\sqrt{17}}{3})$ are critical points.

Step 2. Linearize the system at each critical point.

We have $J = \begin{bmatrix} 2x & 2y \\ -1 & 6y \end{bmatrix}$.

Therefore,

$$J(-3,0) = \begin{bmatrix} -6 & 0 \\ -1 & 0 \end{bmatrix}, J(\tfrac{8}{3}, \tfrac{\sqrt{17}}{3}) = \begin{bmatrix} \frac{16}{3} & \frac{2\sqrt{17}}{3} \\ -1 & 2\sqrt{17} \end{bmatrix} \text{ and } J(\tfrac{8}{3}, -\tfrac{\sqrt{17}}{3}) = \begin{bmatrix} \frac{16}{3} & -\frac{2\sqrt{17}}{3} \\ -1 & -2\sqrt{17} \end{bmatrix}.$$

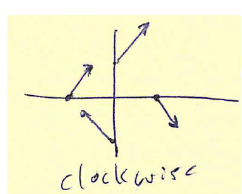
Step 3. Find the eigenvalues and eigenvectors for each critical point.

$(-3,0)$	$-6, \begin{bmatrix} 6 \\ 1 \end{bmatrix}$;	$0, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$;	one attracting, one frozen.
$(\frac{8}{3}, \frac{\sqrt{17}}{3})$	$\approx 6.8 \pm 0.8i$		spiral out, unstable.
$(\frac{8}{3}, -\frac{\sqrt{17}}{3})$	$\approx 5.5, \begin{bmatrix} -13.8 \\ 1 \end{bmatrix}$;	$\approx -8.5, \begin{bmatrix} 0.199 \\ 1 \end{bmatrix}$;	saddle point, unstable.

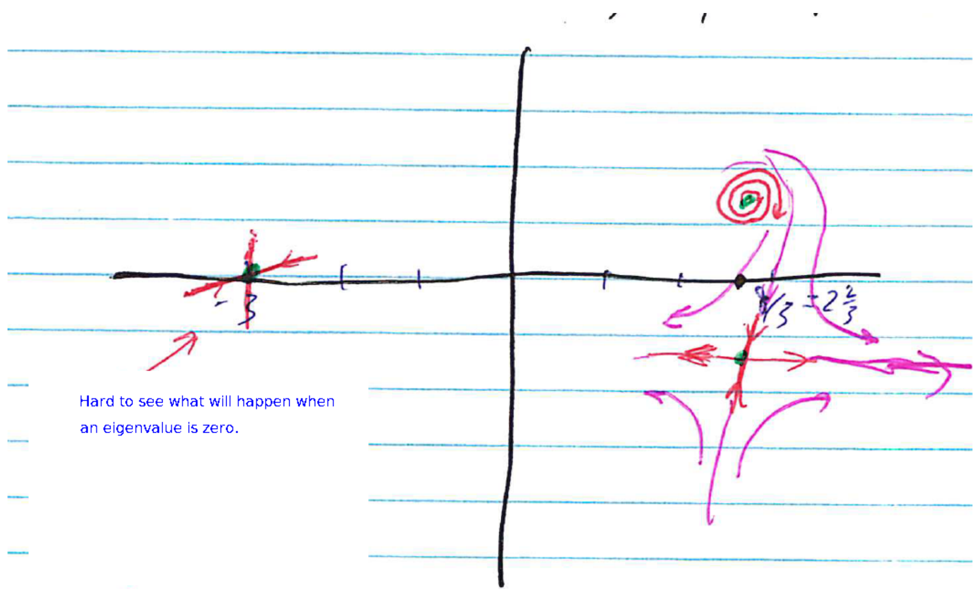
Note: I did not bother with the eigenvectors when the eigenvalues are complex, because they do not tell me that much. I can however check a few points and determine if the spiral is going clockwise or counterclockwise. The linearization is

$$x' = \tfrac{16}{3}x + \tfrac{2\sqrt{17}}{3}y, \quad y' = -x + 2\sqrt{17}y.$$

Plug in $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$. See below. The spiral in this system is clockwise.



Step 4. Draw local patches at the critical points. Try to fit it all together.



Step 5. Compare with computer. This time I only used Maple. The code is on the course website. You should experiment with some other graphing programs.