9.4 Competing (and Cooperating) Species

You should review Section 2.5.

Here we look at models for two species of the form

$$x' = r_1 x\left(1 - \frac{x}{K_1}\right) \pm \alpha_1 xy$$

$$y' = r_2 y\left(1 - \frac{y}{K_2}\right) \pm \alpha_2 xy.$$

If $\alpha_1 = \alpha_2 = 0$, these species do not interact. K_1, K_2 are the carrying capacities studied in 2.5.

The xy terms are "interactions." Each "interaction" might benefit or harm each species.

Example 1

$$x' = x - x^2 - xy$$
$$y' = 2y - y^2 - xy$$

Step 1 Find critical points.

$x' = 0 \Rightarrow x(1-x-y) = 0 \Rightarrow x = 0$ or $x + y = 1$

$y' = 0 \Rightarrow y(2-y-x) = 0 \Rightarrow y = 0$ or $x + y = 2$

Clearly, $(0,0)$ is a cr. pt. So are $(0,+2)$ and $(1,0)$. There are no others since $x+y=1$ and $x+y=2$ can never both be true.

Step 2 Linearize at each cr. pt.

$$\text{Jacobian matrix} = \begin{bmatrix} 1-2x-y & -x \\ -y & 2-2y-x \end{bmatrix}$$

$(0,0)$	$(0,2)$	$(1,0)$
$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix}$	$\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$

Step 3

$(0,0)$	$(0,2)$	$(1,0)$
e. values are $\lambda = 1, 2$.	e. values are $\lambda = -1, -2$	e. values are $\lambda = 1$, $\lambda = -1$.
e. vectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	e vectors are $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	e. vectors are $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
unst. node (repeller)	asy. st. node (attractor)	saddle

Step 4 Put the pieces together.

x goes extinct in the presence of y.

Example 2 $x' = 2x - 2x^2 + xy$
$y' = 5y - 3y^2 + \frac{1}{2}xy$

Step 1 Find c. pts.

$x'=0 \Rightarrow x(2-2x+y)=0 \Rightarrow x=0$ or $2x-y=2$
$y'=0 \Rightarrow y(5-3y+\frac{1}{2}x)=0 \Rightarrow y=0$ or $x-6y=-10$

Thus, $(0,0)$, $(0,\frac{5}{3})$, $(1,0)$ are cr. pts.

This time the two equations $2x-y=2$ and $x-6y=-10$ do have a common solution. You can check $(2,2)$ is it.

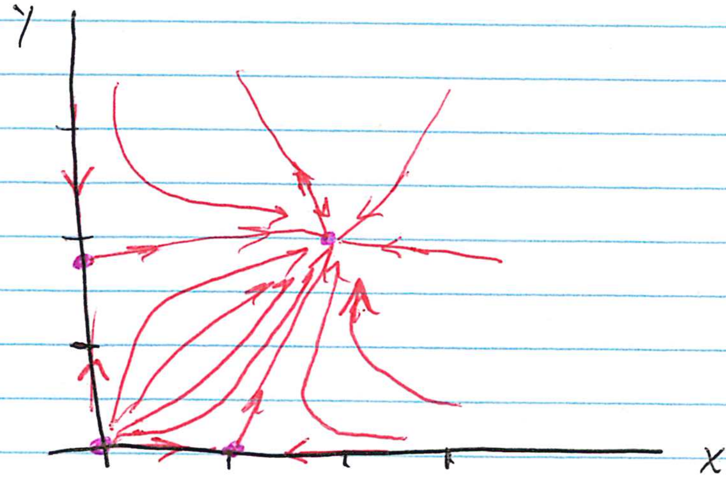
Thus, we have four critical pts.

$$\boxed{J = \begin{bmatrix} 2-4x+y & x \\ \frac{1}{2}y & 5-6y+\frac{1}{2}x \end{bmatrix}}$$

Steps 2&3 Linearize and find e.values and e.vectors.

C.pt.	Jacobian	eigenvalues and eigenvectors	
$(0,0)$	$\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$	$2, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $5, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.	unst. node (repeller)
$(0,5/3)$	$\begin{bmatrix} \frac{11}{3} & 0 \\ \frac{5}{6} & -5 \end{bmatrix}$	$\frac{11}{3} \begin{bmatrix} 52 \\ 5 \end{bmatrix}$, $-5 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	saddle
$(1,0)$	$\begin{bmatrix} -2 & 1 \\ 0 & \frac{11}{2} \end{bmatrix}$	$-2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\frac{11}{2} \begin{bmatrix} 2 \\ 15 \end{bmatrix}$	saddle
$(2,2)$	$\begin{bmatrix} -4 & 2 \\ 1 & -6 \end{bmatrix}$	$-5+\sqrt{3} \approx -3.27$ $\begin{bmatrix} 2 \\ -1+\sqrt{3} \end{bmatrix}$, $-5-\sqrt{3}$, $\begin{bmatrix} 2 \\ -1-\sqrt{3} \end{bmatrix}$	stable node attractor.

Step 4 Put pieces together



x and y each do better when the other is present.