9.5 Predator-Prey Problems and Lotka-Volterra Equations.

Ex 1 Consider
$$x' = x - \tfrac{1}{2}xy$$
$$y' = -\tfrac{3}{4}y + \tfrac{1}{4}xy.$$

Perhaps $x(t)$ = bunnies and $y(t)$ = foxes.
In this model when a fox and a bunny interact, it does not go well for the bunny. If there are no bunnies the fox population collapses. This model does incorporate a finite carrying capacity for the bunnies.

Find the critical points. Clearly $(0,0)$ is a cr. pt.

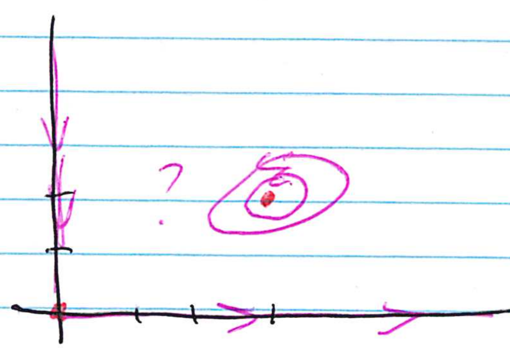
$$x - \tfrac{1}{2}xy = 0 \Rightarrow \quad x = 0 \text{ or } y = 2.$$
$$-\tfrac{3}{4}y + \tfrac{1}{4}xy = 0 \Rightarrow \quad y = 0 \text{ or } x = 3.$$

Neither $(0,2)$ or $(3,0)$ are cr pts. But $(3,2)$ is a cr. pt.

Linearize $J = \begin{bmatrix} 1 - \frac{1}{2}y & -\frac{1}{2}x \\ \frac{1}{4}y & -\frac{3}{4} + \frac{1}{4}x \end{bmatrix}$

$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -3/4 \end{bmatrix}$ saddle

$J(3,2) = \begin{bmatrix} 0 & -3/2 \\ \frac{1}{2} & 0 \end{bmatrix}$ E. values are $\pm \frac{\sqrt{3}}{2} i$, pure imaginary.
Center.
You can check it is ccw.



But, this an approx.
What is really happening near $(3,2)$?

This is a rare case where we can find formulas for solution curves.

If our initial value is $(0, F)$, then the solution is

$$x(t) = 0$$
$$y(t) = F e^{-\frac{3}{4}t}.$$

If our init value is $(B, 0)$, then the solution is

$$x(t) = Be^{t}$$
$$y(t) = 0.$$

If our init. value is elsewhere we can do the following.

By the chain rule $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\frac{3}{4}y + \frac{1}{4}xy}{x - \frac{1}{2}xy}$

$$= \frac{-3y + xy}{4x - 2xy} = \frac{y(x-3)}{x(4-2y)}.$$ This is separable.

$$\Rightarrow \int \frac{4-2y}{y}\, dy = \int \frac{x-3}{x}\, dx$$

$$\Rightarrow \int \frac{4}{y} - 2\, dy = \int 1 - \frac{3}{x}\, dx$$

$$\Rightarrow 4 \ln y - 2y = x - 3 \ln x + C.$$

The textbook claims (pg 547, 10th Ed) that this is a closed curve. I'll show this graphically and outline how it can be proven using topology. I'll do this for the init. cond. $x(0) = 1$, $y(0) = 1$.

At $(1,1)$ the last eq. becomes $-2 = 1 + C$, so $C = -3$. Thus,

$$4\ln y - 2y = x - 3\ln x - 3.$$

$$\Rightarrow \quad \ln y^4 + \ln x^3 = x + 2y - 3$$

$$\Rightarrow \quad \ln y^4 x^3 = x + 2y - 3$$

$$\Rightarrow \quad y^4 x^3 = e^{x+2y-3}.$$

Let $f(x,y) = y^4x^3$ and $g(x,y) = e^{x+2y-3}$, $x \geq 0, y \geq 0$.

Consider the two surfaces $z = f(x,y)$ and $z = g(x,y)$. The intersection of their graphs is our solution curve. See the video post below to see their graphs and their intersection.

The rest of this is optional. I'll use some tools from differential topogoly.

Along the coordinate axes $f(x,y) = 0 < g(x,y)$.

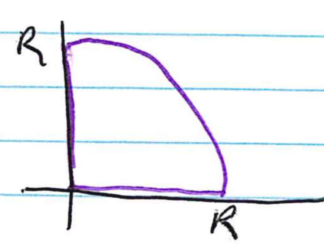
Claim I claim $\exists R > 0$ s.t. for $\sqrt{x^2+y^2} \geq R$, $x \geq 0, y \geq 0$, we also have

$$f(x,y) < g(x,y).$$

So, if they intersect, this must occur inside the quarter-disk

$$D = \{(r,\theta) \in \mathbb{R}^2 \mid 0 \leq \theta \leq \tfrac{\pi}{2},\ 0 \leq r \leq R\}.$$

theta



Pf of Claim: We use polar coordinates.

$$f(r,\theta) = r^7 \cos^3\theta \sin^4\theta.$$
$$g(r,\theta) = e^{r(\cos\theta + 2\sin\theta) - 3}.$$

Let $\alpha = \max\{\cos^3\theta \sin^4\theta \mid 0 \le \theta \le \frac{\pi}{2}\} \approx 0.7 < 1.$

Notice $1 = \min\{\cos\theta + 2\sin\theta \mid 0 \le \theta \le \frac{\pi}{2}\}.$

Thus $f(r,\theta) \le r^7\alpha$ and $e^{r-3} \le g(r,\theta)$, for $0 \le \cancel{\pi}\theta \le \frac{\pi}{2}$.

By L'Hospital's Rule

$$\lim_{r\to\infty} \frac{\alpha r^7}{e^{r-3}} = 0.$$

Thus, $\exists R > 0$ s.t. $r \ge R \Rightarrow \frac{\alpha r^7}{e^{r-3}} < 1.$

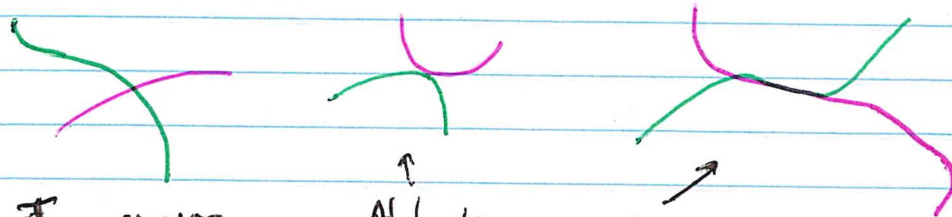
Thus, $r \ge R \Rightarrow \alpha r^7 < e^{r-3} \Rightarrow f(r,\theta) < g(r,\theta),\ 0 \le \theta \le \frac{\pi}{2}.$ $\square$

We can see this intersection is not empty since $f(3,2) = 432 > g(3,2) = e^4 \approx 54.6$

Now, the two surface parts over D are closed bounded sets by a theorem from topology that says the continuous image of a closed bounded set in $\mathbb{R}^2$ is a closed bounded set. Another theorem in top. says the intesection of closed bounded sets in $\mathbb{R}^3$ is a closed bounded set.

Under these conditions Sard's Theorem from differential topology says that if two surfaces intersect transversally the result is one or a finite number of simple closed curves.

What does transverse mean?



Transverse   Not transverse.

We can prove that our two surfaces meet transversally by showing that their grandient vectors never line up, that they are never parallel.

Let $F(x,y,z) = x^3y^4 - z$ and $G(x,y,z) = e^{x+2y-3} - z$.

Then the level surfaces $F(x,y,z)=0$ and $G(x,y,z)=0$ are the surfaces $z=f(x,y)$ and $z=g(x,y)$, resp. Their gradient vectors ar

$$\nabla F = \langle 3x^2y^4, 4x^3y^3, -1\rangle, \quad \nabla G = \langle e^{x+2y-3}, 2e^{x+2y-3}, -1\rangle.$$

We will show that there is no point on $f(x,y) = g(x,y)$ where $\nabla F = \lambda \nabla G$, for any λ. That is the soluton set to

$$\begin{array}{ll} \text{(i)} & 3x^2y^4 = \lambda e^{x+2y-3} \\ \text{(ii)} & 4x^3y^3 = 2\lambda e^{x+2y-3} \\ \text{(iii)} & -1 = -\lambda \\ \text{(iv)} & x^3y^4 = e^{x+2y-3} \end{array}$$

is empty. Clearly, $\lambda=1$, if there is a solution. Then by (iv), $x \neq 0$ and $y \neq 0$. Thus,

$$3x^2y^4 = 2x^3y^3 = x^3y^4 \Rightarrow 3y = 2x = xy.$$

But $2x = xy \Rightarrow y=2$, and $3y = xy \Rightarrow x=3$. But $f(3,2) \neq g(3,2)$. Hence, the surfaces meet transversally.

So, the intersection is a finite number of simple closed curves. It can be shown that there is only one, but even if there were several, only one would be our unique solution curve.

See https://en.wikipedia.org/wiki/Transversality_(mathematics)