Section 9.6 Lyapunov's Second Method

Def Let D be an open set in $\mathbb{R}^2$ with $(0,0)$ in D.
Let $V: \mathbb{R}^2 \to \mathbb{R}$, with $V(0,0) = 0$.

1. V is positive definite on D if $V(x,y) > 0$ on $D - \{(0,0)\}$.
2. V is positive semi-definite on D if $V(x,y) \geq 0$ on D.
3. V is negative definite on D if $V(x,y) < 0$ on $D - \{(0,0)\}$.
4. V is negative semi-definite on D if $V(x,y) \leq 0$ on D.

Ex $V(x,y) = x^2 + y^2$ is pos. def. on $\mathbb{R}^2$.

$V(x,y) = \sin(x^2+y^2)$ is pos. def. on $D = \{(x,y) \mid x^2+y^2 < \pi/2\}$

$V(x,y) = (x+y)^2$ is pos. semi-def on $\mathbb{R}^2$.

Thm (9.6.4) Let $V(x,y) = ax^2 + bxy + cy^2$. Let $D = \mathbb{R}^2$.

V is pos. def. on D iff $a > 0$ and $4ac - b^2 > 0$.

V is neg. def on D iff $a < 0$ and $4ac - b^2 > 0$.

Pf Use the 2-variable Second Derivative Test.

Thm (9.6.1) Suppose the autonomous system

$$x' = F(x,y)$$
$$y' = G(x,y)$$

has an isolated critical pt. at $(0,0)$. ⓐ Suppose $V(x,y)$ is cont., has cont. first partial derivatives, is pos. def. and that $\dot{V}(x,y) = V_x x' + V_y y'$ is neg. def. all in some open set D containing $(0,0)$. Then $(0,0)$ is asymptotically stable. ⓑ If instead $\dot{V}$ is neg. semi-def., then $(0,0)$ is at least stable.

Thm (9.6.2) Suppose the autonomous system

$$x' = F(x,y)$$
$$y' = G(x,y)$$

has an isolated c. pt. at $(0,0)$. Let D be an open set containing $(0,0)$. Let $V(x,y)$ be cont. and have cont. first partial derivatives in D with $V(0,0)=0$. Suppose in any open disk about the origin $\exists$ a pt at which $V(x,y)$ is pos. (neg.). If $\dot{V}$ is pos. def. (neg def.) on D, then $(0,0)$ is an unstable critical pt.

We will discuss the motivation for these in class.

A standard application is the undamped pendulum. Study Example 2 in the text book.

Ex (#1 in 9.6) $x' = -x^3 + xy^2$ and $y' = -2x^2y - y^3$.
Show $(0,0)$ is the only critical point and that it is asy. st.

Solution Clearly, $(0,0)$ is a c. pt. To show it is the only one, look first at $y' = -2x^2y - y^3$.

$$-2x^2y - y^3 = 0 \iff -y(2x^2+y^2) = 0 \iff y=0 \text{ or } (x=0 \text{ and } y=0).$$

Thus, for any c.p. we know $y=0$. Now look at $x' = -x^3 + xy^2$.
If we are at a c.p. $y=0$ so $x' = -x^3 = 0$. Thus, $x=0$.
Thus, $(0,0)$ is the only c.p.

If we linearize at $(0,0)$, the Jacobian matrix is

$$\begin{bmatrix} -3x^2+y^2 & 2xy \\ -4xy & -2x^2-3y^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ at } (0,0).$$

Both eigenvalues are 0 so it is not helpful.

We try to find a Lyapunov function by letting $V(x,y) = ax^2 + cy^2$.
Then

$$\dot{V} = \frac{dV}{dt} = V_x x' + V_y y' = 2ax(-x^3+xy^2) + 2cy(-2x^2y - y^3)$$

$$= -2ax^4 + 2ax^2y^2 - 4cx^2y^2 - 2cy^4$$

$$= -2(ax^4 + (a-2c)x^2y^2 + cy^4).$$

Let $a=2$ and $c=1$. Then $V = 2x^2 + y^2$ is pos. def.
and $\dot{V}$ is neg. def. By Thm 9.6.1 we know that $(0,0)$
is asy. st.

Ex (#6 in 9.6) A generalization of the undamped pendulum equation is $u''(t) + g(u(t)) = 0$, where $g(0) = 0$, $g(u) > 0$ for $0 < u < k$ (k is some given positive constant) and $g(u) < 0$ for $-k < u < 0$. We are going to show the equalibrium solution is stable.

First, notice $g(u) = \sin(u)$ has the desired properties for $k = \pi$. Other examples this result will apply to are
$u'' + u = 0$, $u'' + 5u = 0$, $u'' + u^3 = 0$, $u'' + (u - u^3) = 0$ $k=1$,
$u'' + \arctan(u) = 0$, $u'' + \tan u = 0$, $k = \pi/2$, $u'' + \frac{u}{1+u^2} = 0$.

Solution ① Convert to a 2x2 system as follows.

Let $x = u$, $y = u'$.

Then $x' = y$ and $y' = -g(x)$.

② Clearly, $(0,0)$ is an isolated c.p.

③ Let $V(x,y) = \frac{1}{2}y^2 + \int_0^x g(s)\,ds \quad (-k < x < k)$.

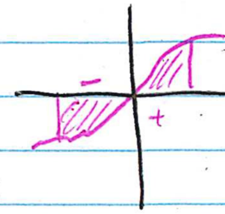
ⓐ We will show $V(x,y)$ is pos. def.
ⓑ We will show $\dot{V}(x,y)$ is neg. semi-def.

By the second part of Thm 9.6.1, $(0,0)$ is a stable c.p.

(a) $V(0,0) = \frac{1}{2}0^2 + \int_0^0 g(s)\,ds = 0$

For any other (x,y), $\frac{1}{2}y^2 \geq 0$.

For $x > 0$, the $\int_0^x g(s)\,ds > 0$



For $x < 0$, the $\int_0^x g(s)\,ds = -\int_x^0 g(s)\,ds$

$= -(-) = + > 0.$

Thus, if $(x,y) \neq (0,0)$, $\int_0^x g(s)\,ds > 0$ if $x \neq 0$

and $\frac{1}{2}y^2 > 0$ if $y \neq 0$.

Thus, $V(x,y) > 0$, for $(x,y) \neq 0$.

(b) $\dot{V} = \frac{dV}{dt} = V_x x' + V_y y' = \ ?$

$V(x,y) = \frac{1}{2}y^2 + \int_0^x g(s)\,ds.$ $\quad x' = y,\ y' = -g(x).$

$V_y = y,\ V_x = g(x).$

Thus, $\dot{V} = gy - yg = 0$, always.

Thus, $(0,0)$ is stable.

Problems 7, 8 and 9 continue these ideas and could be part of a project.