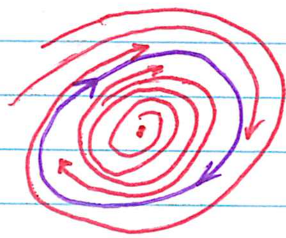
9.7 Periodic Solutions and Limit Cycles

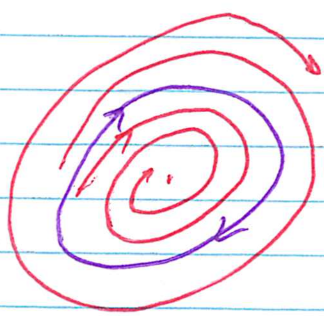
Def A vector function $v(t)$ in $\mathbb{R}^n$ is <u>periodic</u> if for some $T \neq 0$ we have $v(t+T) = v(t)$ for all t. If T is the smallest positive number with this property, we call T the <u>period</u> of $v(t)$.

Note: We normally exclude constant functions.

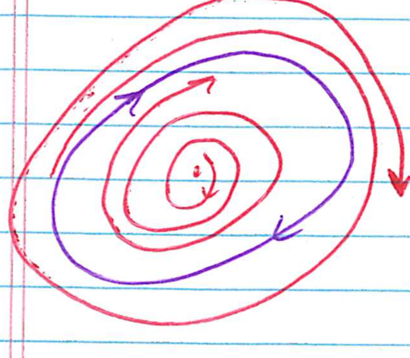
Sometimes differential equations have periodic solutions curves. We call these <u>closed curves.</u> If other solution curves coil toward a periodic solution curve it is called a <u>limiting cycle.</u>



This limiting cycle is said to be <u>asymptotically stable.</u> Also call an <u>attracting periodic orbit.</u>



This periodic solution is unstable. Also called a <u>repelling periodic orbit.</u>



This closed solution curve is <u>semi-stable</u>

<u>Stable</u> but not asym. st.

Ex 1 Consider
$$x' = (4-x^2-y^2)x - 3y$$
$$y' = (4-x^2-y^2)y + 3x. \qquad (\ast)$$

Find critical point(s). Clearly $(0,0)$ is a cr. pt. We will it is the only one. If $x=0$, then $y=0$. If $y=0$, then $x=0$. So we can assume $x \neq 0, y \neq 0$.

$x' = 0 \Rightarrow \quad (4-x^2-y^2) = 3y/x.$

$y' = 0 \Rightarrow \quad (4-x^2-y^2) = -3x/y.$

Thus,
$$\frac{3y}{x} = \frac{-3x}{y} \Rightarrow y^2 = -x^2 \Rightarrow y^2 + x^2 = 0$$
$$\Rightarrow x = y = 0.$$

Thus, $(0,0)$ is the only cr. pt.

Linearize
$$J = \begin{bmatrix} -2x^2 + (4-x^2-y^2) & -2xy - 3 \\ -2xy + 3 & -2y^2 + (4-x^2-y^2) \end{bmatrix}.$$

$$J(0,0) = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}.$$ Eigenvalues are $4 \pm 3i$.
$\Rightarrow$ outward spirial.

But, something interesting is happening when $x^2+y^2=4$. To better see this we will convert to polar coordinates.

$r^2 = x^2+y^2$. $\quad x = r\cos\theta \quad y = r\sin\theta$

$x' = r'\cos\theta - r\sin\theta\,\theta' \quad y' = r'\sin\theta + r\cos\theta\,\theta'$

Thus $(\ast)$ becomes

$$r'\cos\theta - r\sin\theta\,\theta' = (4-r^2)r\cos\theta - 3r\sin\theta$$
$$r'\sin\theta + r\cos\theta\,\theta' = (4-r^2)r\sin\theta + 3r\cos\theta$$

$$\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}\begin{bmatrix} r' \\ \theta' \end{bmatrix} = \begin{bmatrix} (4-r^2)r\cos\theta - 3r\sin\theta \\ (4-r^2)r\sin\theta + 3r\cos\theta \end{bmatrix}$$

Inverse is $\frac{1}{r}\begin{bmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Thus, $\begin{bmatrix} r' \\ \theta' \end{bmatrix} = \frac{1}{r}\begin{bmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} (4-r^2)r\cos\theta - 3r\sin\theta \\ (4-r^2)r\sin\theta + 3r\cos\theta \end{bmatrix}$

(This will simplify!)

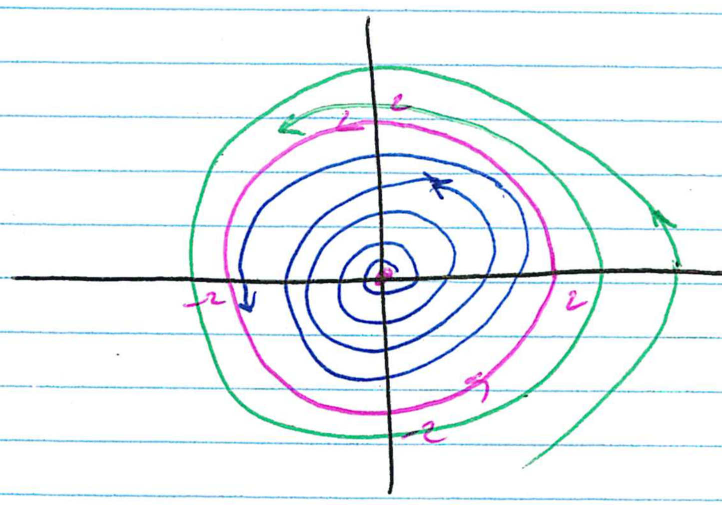
$$= \frac{1}{r}\begin{bmatrix} (4-r^2)r^2c^2 - 3r^2cs + (4-r^2)r^2s^2 + 3r^2cs \\ -(4-r^2)rcs + 3rs^2 + (4-r^2)rcs + 3rc^2 \end{bmatrix}$$

$$= \begin{bmatrix} (4-r^2)r \\ 3 \end{bmatrix}$$

Thus, $r' = (4-r^2)r$, $r' = 0$ when $r = 2$.
$\theta' = 3$. $\longrightarrow$ constant ccw rotation.

Thus, $r = 2$, $\theta = 3t$ is a periodic solution.

For $0 < r < 2$, $r' > 0$, outward.
For $r > 2$, $r' < 0$, inward.
$\Rightarrow$ $r = 2$, $\theta = 3t$ is a limit cycle.



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> with(DEtools):
> phaseportrait([D(x)(t)= (4-x(t)^2-y(t)^2)*x(t) - 3*y(t),
                 D(y)(t) = (4-x(t)^2-y(t)^2)*y(t) + 3*x(t)],
                [x(t),y(t)], t=0..3, [[x(0)=2,y(0)=0]],x=-3..3,y=
-3..3,color=black,linecolor=red);
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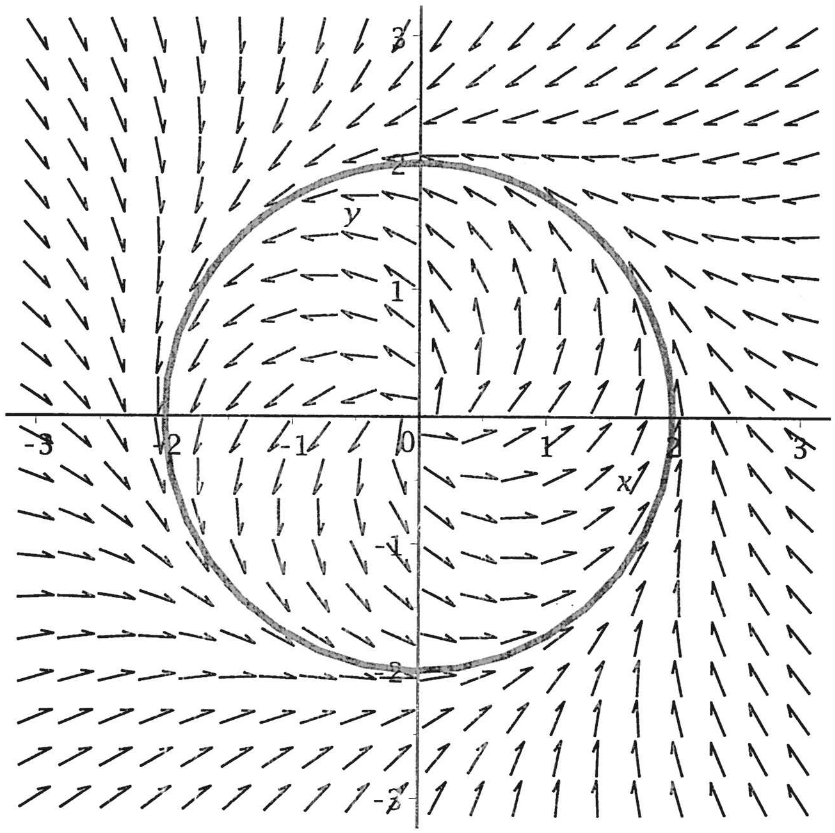


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>
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Ex 2 When doing Example 1 on the computer I entered the following by mistake. The graphs where cool so I decided to play with it.

$$x' = 4 - x^2 - y^2 - 3y$$
$$y' = 4 - x^2 - y^2 + 3x$$

The phase portrait is on the next page. You can see there is a saddle near (2, -2) and perhaps a center near (-1, 1). I was curious if the closed orbits were really closed loops or if this was just a computer artefact.

The two critical points where at $\left(\frac{3-\sqrt{41}}{4}, -\frac{3-\sqrt{41}}{4}\right)$ and $\left(\frac{+3+\sqrt{41}}{4}, -\frac{3+\sqrt{41}}{4}\right)$ or numerically about $(-0.85, 0.85)$ and $(2.351, -2.351)$. The linearization showed the first had pure imaginary eigenvalue, $\pm\sqrt{3\sqrt{41}}\, i$, while the second was a saddle with eigenvalues $\pm\sqrt{3\sqrt{41}}$.

I decided to check the initial condition $x(0)=0$, $y(0)=0$, to see if I could show it really was a closed loop.

I tried to solve $\frac{dy}{dx} = \frac{4-x^2-y^2+3x}{4-x^2y^2-3y}$, but failed.

Then used dsolve, and after some fiddling got an implicit solution of

$$-\ln 64 + 3\ln\left(\begin{matrix}4-x^2-y^2\\ \cancel{x^2+y^2-4}\end{matrix}\right) + 2y - 2x = 0.$$

I took the derivative and solved for $\frac{dy}{dx}$ to verify it was correct.

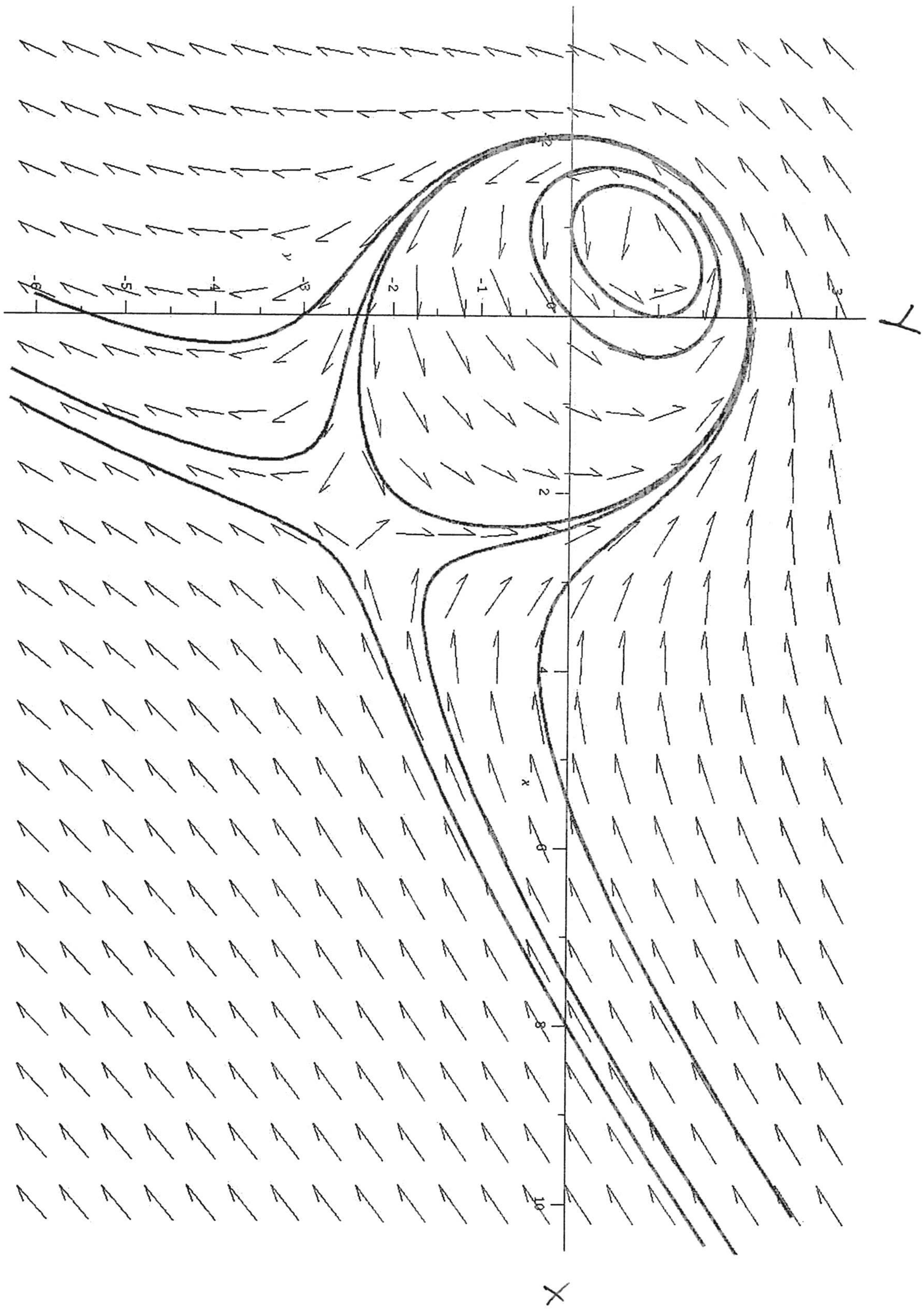
Then I converted this to

$$\left(\frac{(4-x^2-y^2)^3}{64}\right) = e^{2(x-y)}.$$

The LHS is zero on the circle of radius 2, center $(0,0)$ and is only positive inside this circle. The RHS is always positive, so any intersection the two surfaces

$$z = \frac{(4-x^2-y^2)^3}{64} \quad \text{and} \quad z = e^{2(x-y)}$$

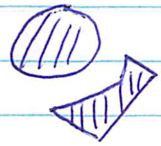
occurs inside this circle. They do meet at $(0,0)$, so their intersection is not empty. You can check their gradients and show they meet transversally. By Sard's Theorem the intersection is a finite number of closed loops. In fact there is only one, but if there were more we would only use the one containing $(0,0,1)$.



Y
X

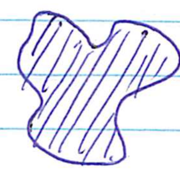
Here are three theorems. Their statements can seem a bit convoluted, but they are easy to visualize. They only apply to vector fields in $\mathbb{R}^2$.

Def A subset or domain D in $\mathbb{R}^2$ is <u>simply connected</u> if it is connected and has no holes.

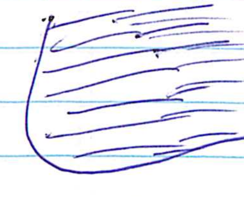


Not connected

Connected, but not simply connected



simply connected.

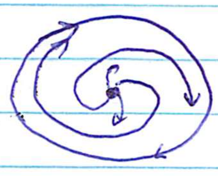


simply connected

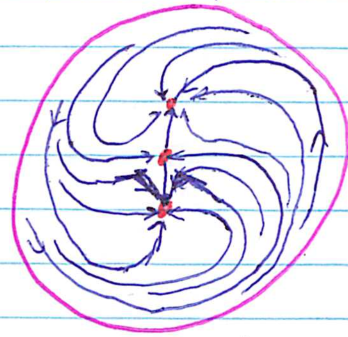
Let $\frac{dx}{dt} = F(x,y)$, $\frac{dy}{dx} = G(x,y)$ and assume they have continuous first partial derivates in a simply connected domain $D \subset \mathbb{R}^2$.

Thm 9.7.1 If there is a closed solution curve γ in D, then the system will have a critical point (at least one) inside the region bounded by γ. If γ encloses only one critical point, the critical point cannot be a saddle.

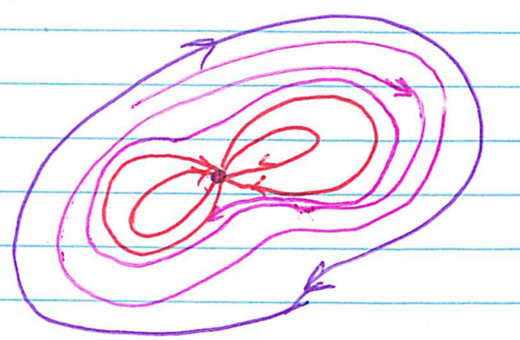
We will not prove this, but it is a special case of the Poincaré-Hopf Index Theorem that we will discuss later. Here are some pictorial examples.



~~reller~~ repeller



a saddle, also two attractors



A nonlinearizable node.

Thm 9.7.2 If $F_x + G_y$ has the same sign throughout D (i.e. it is never zero) then there is no closed solution curve in D.

The proof uses Green's Theorem from Calc III. See Problem #13 in this section. It is extra credit!

Here ~~two~~ some simple examples.

Ex $x' = 7x + y$
$y' = 2y.$

Then $F_x + G_y = 7 + 2 = 9$ is always positive in all of $\mathbb{R}^2$. Thus, there are no closed solution curves.

Ex $x' = x^3 + x + \sin(y) \cdot e^y$
$y' = 2y + x^4 + e^x.$

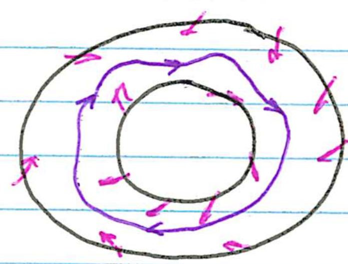
Then $F_x + G_y = 3x^2 + 1 + 2$ is always positive in all of $\mathbb{R}^2$. Thus there ~~is~~ are no ~~solu~~ closed solution curves.

Thm 9.7.3

The Poincaré-Bendixson Theorem

Def A region of $\mathbb{R}^2$ that is connected, but has just one hole, ~~is said to be an~~ and has two closed curves as its boundary, is said to be an annular domain.

The Poincaré-Bendixson Theorem implies the following. Suppose $x' = F(x,y)$, $y' = G(x,y)$ where F and G has cont. partial derivatives in ~~an~~ closed annular domain A. If there are no critical pts in A and if any solution curve that ~~is~~ ~~has~~ meets A stays in A for a future times, then A contains a periodic solution curve.



The book's statement of the PB-Thm is more general in that the domain could have several holes. The PB Thm is still true if all orbit meeting the boundary exit the domain. There could more than one periodic solution curve in the domain.