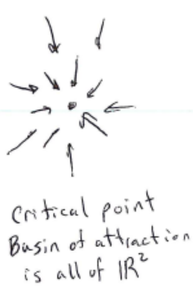
Section 9.1

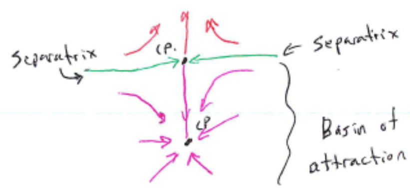
This section is just a review of Chapter 7. Read it on your own. Table 9.1.1 and Problems #20 and 21 give a good summary of the stability types of nodes.

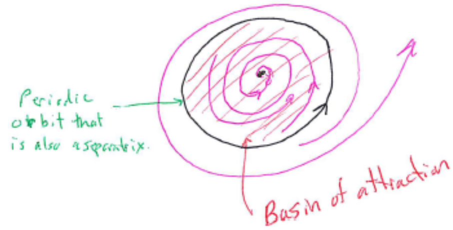
Section 9.2

The key ideas are ...

- **Critical points.** Points $(x_1, x_2, \ldots, x_n)$ where all the derivatives are zero, $x_1' = x_2' = \cdots x_n' = 0$.
- **Basin of attraction** of a critical point or a periodic orbit. It is the set of points for which the solution curve converges to the given critical point or periodic solution.
- **Separatrix.** A solution curve that is part of the boundary of a basin of attraction.







Example (# 7 in Problems). Consider the system of equations below.

$$\frac{dx}{dt} = 2x - x^2 - xy$$

$$\frac{dy}{dt} = 3y - 2y^2 - 3xy$$

We will find the critical points and then use a computer to draw direction fields and try to understand what is happening near each of the critical points.

$$2x - x^2 - xy = 0 \implies x(2 - x - y) = 0 \implies x = 0 \text{ or } x + y = 2.$$

$$3y - 2y^2 - 3xy = 0 \implies y(3 - 2y - 3x) = 0 \implies y = 0 \text{ or } 3x + 2y = 3.$$

If $x = 0$, then $y = 0$ or $y = 3/2$. Thus, $(0,0)$ and $(0,3/2)$ are critical points.

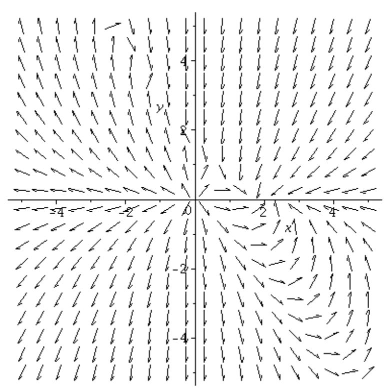
If $y = 0$, then $x = 0$ or $x = 2$. Thus, $(2,0)$ is also a critical point.

The only other case is $x + y = 2$ and $3x + 2y = 3$. This gives a fourth critical point, $(-1,3)$.

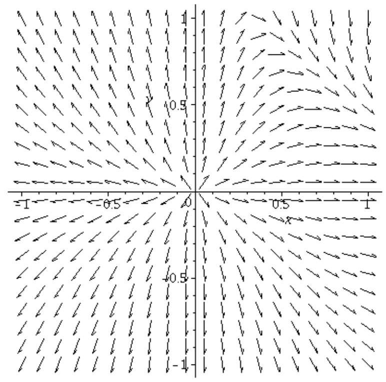
Thus, the critical points are $(0,0)$, $(0,3/2)$, $(2,0)$ and $(-1,3)$.

Here is something else worth noting. If $y = 0$, then $y' = 0$. So, if we start on the x-axis we will stay on the x-axis. Likewise, if $x = 0$, then $x' = 0$. So, if we start on the y-axis we will stay on the y-axis.

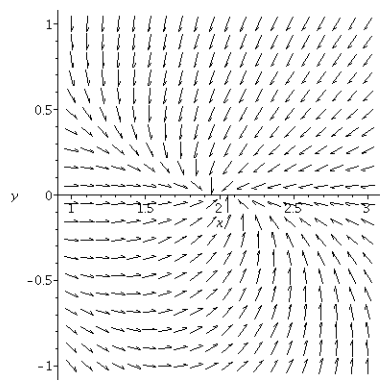
Now we start plotting. The first plot is of the direction field for our system over $-5 \le x \le 5$ and $-5 \le y \le 5$. We can see the critical point at $(0,0)$ is a repeller. The critical point at $(2,0)$ is an attractor - that is, it is asymptotically stable. Although it looks kind of like a spiral, we know the x-axis is never crossed by solution curves. So, maybe it is an improper node. It is really hard to tell what is going on at the other two critical points.



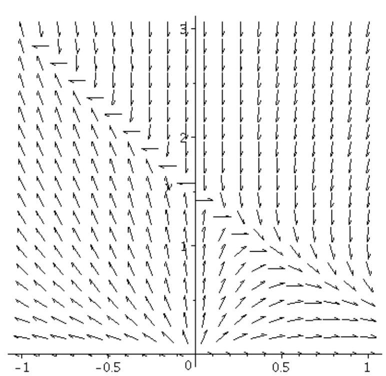
Now we will "zoom in" on each critical point. The plot below is for $-1 \le x \le 1$ and $-1 \le y \le 1$. We can see even more clearly that $(0,0)$ is a repeller. In Section 9.3 we will learn how to verify such a claim by finding a linear system that approximates our system in a neighborhood of the critical point.



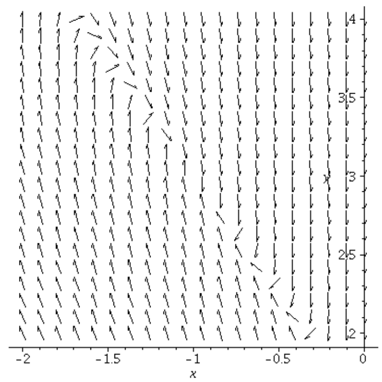
The next plot is over $1 \le x \le 3$ and $-1 \le y \le 1$. Now we get a better view of the direction field near the critical point $(2, 0)$. It looks like an attracting improper node. A linear system that approximates it would have a repeated negative eigenvalue.



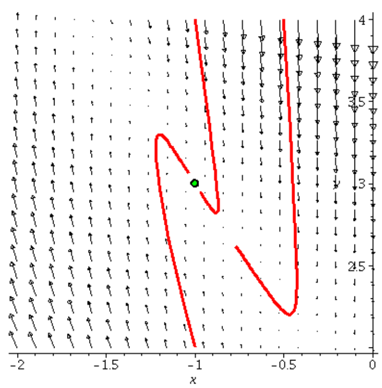
Next we take a closer look at the neighborhood of the critical point at $(0, 3/2)$ on the y-axis. The plot below is over $-1 \le x \le 1$ and $0 \le y \le 3$. It looks like we have a saddle point. Nearby points on the y-axis are drawn toward this critical point. And it seems there is a repelling direction at about a 45 degree angle to the y-axis.



Finally, we look at a neighborhood of the critical point $(-1, 3)$. The plot below is for $-2 \le x \le 0$ and $2 \le y \le 4$. It still looks really weird.



I used some more sophisticated tools to get a better view of the behavior near $(-1, 3)$ for the plot below. This critical point is asymptotically stable but one eigenvalue is extremely small.



Extra Credit!

1. Once we cover Section 9.3, find the linearization at each critical point and find the eigenvalues. Do they match with the discussion above?

2. Below is the Maple code used for first and last plots above. For extra credit, redo these plots using matlab or some other program.

First load packages used for plots.

```
> with(DEtools):with(plots):with(plottools):
```

Code for first plot.

```
> dfieldplot([D(x)(t)=2*x(t)-x(t)^2-x(t)*y(t), D(y)(t)=3*y(t)-2*y(t)^2-3*x(t)*y(t)],[x(t),y(t)],
  t=0..1,x=-5..5,y=-5..5,color=black);
```

Code for last plot.

```
> cp:=disk([-1,3],0.02,color=green):
> vectorfield:=fieldplot([2*x-x^2-x*y,3*y-2*y^2-3*x*y],x=-2..0,y=2..4, arrows=slim, anchor=tail,
  fieldstrength=maximal(2),grid=[20,20]):
> solutioncurves:=phaseportrait([D(x)(t)=2*x(t)-x(t)^2-x(t)*y(t), D(y)(t)=3*y(t)-2*y(t)^2-3*x(t)*y(t)],
  [x(t),y(t)],t=0..3,[[x(0)=-1,y(0)=4],[x(0)=-1,y(0)=2],[x(0)=-1/2,y(0)=4]],x=-2..0,y=2..4,
  linecolor=red,arrows=none):
> display(vectorfield,solutioncurves,cp);
```

Example (#20 in Problems).

Find the solution to $\begin{array}{rcl} x' & = & -x+y \\ y' & = & -x-y, \end{array}$ using equation (17) on page 515.

Solution.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-x-y}{-x+y} = \frac{x+y}{x-y}.$$

We will use the method given in #29 of §2.2.

Let $v = \frac{y}{x}$. Then $\frac{x+y}{x-y} = \frac{1+\frac{y}{x}}{1-\frac{y}{x}} = \frac{1+v}{1-v}$.

Also, since $y = xv$ we have $\frac{dy}{dx} = v + x\frac{dv}{dx}$; we are regarding $v = \frac{y(x)}{x}$ as an implicit function of x.

Now,

$$v + x\frac{dv}{dx} = \frac{1+v}{1-v} \implies x\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v^2}{1-v}.$$

Thus, this equation is **separable**. We may write it as

$$\int \frac{1-v}{1+v^2}\,dv = \int \frac{1}{x}\,dx = \ln|x| + C.$$

Now

$$\int \frac{1-v}{1+v^2}\,dv = \int \frac{1}{1+v^2}\,dv - \int \frac{v}{1+v^2}\,dv.$$

$$= \arctan v - \frac{1}{2}\ln(1+v^2) + C.$$

(For the second integral use $u = 1 + v^2$.)

We conclude that

$$\arctan\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(1 + \frac{y^2}{x^2}\right) = \ln|x| + C.$$

Or,

$$\arctan\left(\frac{y}{x}\right) - \ln\left(|x|\sqrt{1 + \frac{y^2}{x^2}}\right) = C.$$

Or,

$$\arctan\left(\frac{y}{x}\right) - \ln\sqrt{x^2 + y^2} = C$$

Given an initial condition, we could solve for C. But, notice something. The form of this equation is begging us to think about polar coordinates, since

$$r = \sqrt{x^2 + y^2} \;\;\&\;\; \theta = \arctan\left(\frac{y}{x}\right).$$

Thus, we have

$$\theta - \ln r = C.$$

Or,

$$r = e^{\theta - C} = Ce^{\theta} \text{ (new } C\text{)}.$$

The graph of $r = Ce^{\theta}$ is a spiral.

If we go back to the original problem, we can see that it is actually linear.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x - y \\ -x + y \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

If you check, you will find that the eigenvalues are $1 \pm i$. So, the solution curves are indeed spirals.

In fact all the 2×2 linear systems can be solved in this way. If

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{cx + dy}{ax + by} = \frac{c + dv}{a + bv},$$

where $v = \frac{y}{x}$. Using $\frac{dy}{dx} = v + x\frac{dv}{dx}$ we can show that the equation is separable.

Occasionally, this method can be applied to nonlinear 2×2 systems, but this is rare.