Surfaces

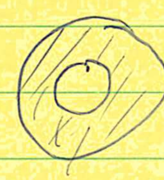
Orientable, compact (closed and bounded) surfaces without boundary are classified up to homeomorphism as follows...

$T^2 \# T^2 \# T^2$

S^2 or the 2-sphere — T^2 or the torus — $T^2 \# T^2$ or the double torus — etc etc.

Orientable, compact surfaces with boundary can be constructed from any of these by removing the interiors of a finite number of disjoint closed disks.

 $S^2 -$ open disk = closed disk 

 $S^2 -$ two open disks = annulus 

Any orientable, compact surface without boundary is homeomorphic to a polyhedron. If the polyhedron happens to have all faces triangles, it is called a triangulation of the surface.

The Euler characteristic of a polyhedron is defined to be $V - E + F$ where

V = # of vertices

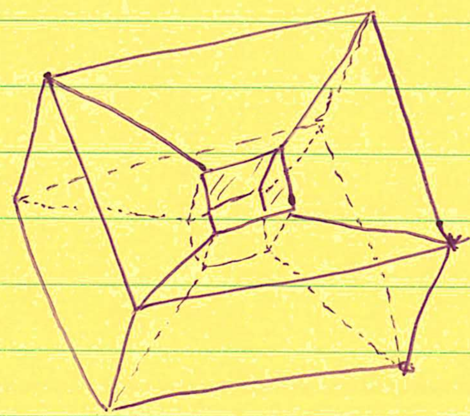
E = # of edges

F = # of faces

Examples:

$V - E + F = 4 - 6 + 4 = 2$

$V - E + F = 8 - 12 + 6 = 2$

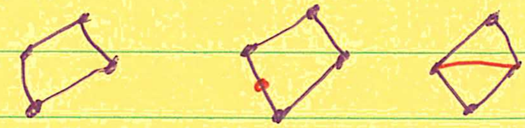


$V - E + F = 16 - 32 + 16 = 0$

polyhedron homeomorphic to a torus.

Fact The Euler characteristic of a polyhedron depends only on the surface it is homeomorphic to.

Rough idea behind proof:



no change when adding a vertex b/c edges increase.

no change when adding an edge b/c faces increase.

The Greek letter χ (chi) denotes the Euler characteristic of a surface.

$$\chi(S^2) = 2, \quad \chi(T^2) = 0.$$

In general $\chi(\underbrace{T^2 \# \cdots \# T^2}_{n \text{ tori}}) = 2 - 2n$

To see this, take any surface and add a "handle"

Remove two faces

$$F_{new} = F_{old} - 2 + 3$$
$$E_{new} = E_{old} + 3$$
$$V_{new} = V_{old}.$$
$$F_{new} - E_{new} + V_{new} = \chi_{old} - 2.$$