**Supplement to 9.7**
**Equation's of Liénard and van der Pol**

Liénard's equation is

$$\tfrac{d^2u}{dt^2} + f(u)\tfrac{du}{dt} + g(u) = 0. \qquad (*)$$

We assume throughout that $f$ and $g$ have continuous first derivatives. Suppose for now that $\gamma = f(u)$ is a positive constant. If $g(u) = ku$ for $k > 0$, this is a model for a mass on a spring. If $g(u) = a\sin u$, this is a model for a pendulum. Thus, Liénard's equation is a generalization of these.

We can convert $(*)$ to a 2x2 system of first order equations. Let $x(t) = u(t)$ and $y(t) = u'(t)$. Now, $x'(t) = y(t)$ and $y'(t) = u''(t) = -f(u)u' - g(u) = -f(x)y - g(x)$. Or,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -f(x)y - g(x) \end{bmatrix}. \qquad (**)$$

**Theorem.** Assume $g(0) = 0$. Then $(0,0)$ is an almost linear critical point of $(**)$. If $f(0) > 0$ and $g'(0) > 0$, then $(0,0)$ is asymptotically stable. If $f(0) < 0$ and $g'(0) < 0$, then $(0,0)$ is unstable.

*Proof.* It is immediate that for $(x,y) = (0,0)$ we have $x' = 0$ and $y' = 0$. To show the system is almost linear we rewrite the equation for $y'$ using Taylor's theorem applied to $f$ and $g$.

$$g(x) = g(0) + g'(0)x + r_g(x)$$

$$f(x) = f(0) + f'(0)x + r_f(x)$$

where $r_g(x)/x$ and $r_h(x)/x$ go to zero as $x \to 0$. Note, that this implies $r_g(x)$ and $r_h(x)$ go to zero as $x \to 0$. We have,

$$y' = -f(0)y - f'(0)xy - r_f(x)y - g'(0)x - r_g(x) = -g'(0)x - f(0)y + (-f'(0)xy - r_f(x)y - r_g(x)).$$

Then

$$\frac{-f'(0)xy - r_f(x)y - r_g(x)}{r} = -f'(0)r\cos\theta\sin\theta - r_f(x)\sin\theta - \frac{r_g(x)}{r}.$$

Taking the limit as $r \to 0$, the first two terms go to zero and

$$\frac{|r_g(x)|}{r} \le \frac{r_g(x)}{|x|} \to 0.$$

Thus, the system is almost linear.

Now assume $f(0) > 0$ and $g'(0) > 0$. The Jacobian is

$$J = \begin{bmatrix} 0 & 1 \\ -g'(0) & -f(0) \end{bmatrix}.$$

Let $p = -g'(0)$ and $q = -f(0)$. Then $p < 0$ and $q < 0$. Now,

$$|J - \lambda I| = \lambda^2 - q\lambda - p.$$

Thus,

$$\lambda = \frac{q \pm \sqrt{q^2 + 4p}}{2}.$$

If $q^2 + 4p < 0$, then $\operatorname{Re} \lambda = \frac{q}{2} < 0$ and $(0, 0)$ is an asymptotically stable spiral.

If $q^2 + 4p = 0$, then $\lambda = \frac{q}{2} < 0$ is a repeated root and thus. $(0, 0)$ is an asymptotically stable improper node.

If $q^2 + 4p > 0$, then $\sqrt{q^2 + 4p} < q$. Thus, both values of $\lambda$ are negative and $(0, 0)$ is an asymptotically stable node.

The analysis for the case $f(0) < 0$ and $g'(0) < 0$ is similar. □

In the next theorem we make different assumptions on $f$ and $g$ that enable us to show the existence of a period orbit.

**Theorem.** Assume in $(**)$ that $f$ is even and $g$ is odd. Assume $g(x) > 0$ for $x > 0$. Let

$$F(x) = \int_0^x f(s)\, ds \quad \& \quad G(x) = \int_0^x g(s)\, ds.$$

Suppose, $G(x) \to \infty$ and $F(x) \to \infty$ as $x \to \infty$. Suppose further that there is a value $x_o > 0$ such that $F(x) < 0$ for $0 < x < x_o$, $F(x) > 0$ for $x > x_o$, and $F(x)$ is increasing for $x > x_o$.

Then equation $(*)$ has a unique periodic solution and in the phase portrait for equation $(**)$ all other trajectories converge to the corresponding closed solution curve as $t \to \infty$, expect of course for the critical point $(0, 0)$.

**Example.** Van der Pol's equation is

$$\tfrac{d^2u}{dt^2} - \mu(1 - u^2)\tfrac{du}{dt} + u = 0,$$

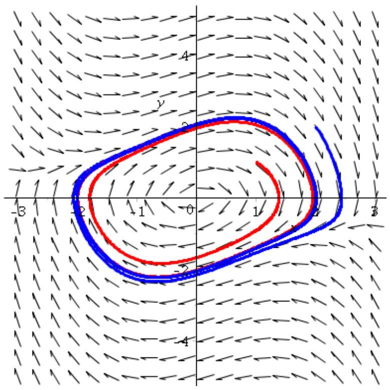
where $\mu > 0$. It is easy to verify the conditions are satisfied. Clearly, $f(u) = -\mu(1 - u^2)$ is even, $g(u) = u$ is odd, and both have continuous derivatives.

$$G(x) = \int_0^x s\,ds = \tfrac{1}{2}x^2,$$

goes to infinity as $x$ does.

$$F(x) = \int_0^x -\mu(1-s^2)\,ds = -\mu\left(x - \tfrac{x^3}{3}\right) = \mu u\left(\tfrac{x^2}{3} - 1\right).$$

Thus, $F(x) < 0$ for $0 < x < \sqrt{3}$ and $F(x) > 0$ and increasing for $x > \sqrt{3}$. Further, $F(x) \to \infty$ as $x \to \infty$. Thus, the previous theorem applies. A phase portrait for $\mu = 0.5$ is show below.



We will not prove this theorem. It was proven by Levinson and Smith in their article *A general equation for relaxation Oscillations* in the *Duke Journal of Mathematics*, Vol. 9, 1942, pages 382–403. They use the Poincaré-Bendixson theorem to show a periodic solution exists, and then go on to show other trajectories converge to it. A proof for the special case of the van der Pol equation can be found in the textbook *Differential Equations, Dynamical Systems, and Linear Algebra*, by Hirsch and Smale, pages 218–225.

Here are some additional examples you can experiment with. Add some parameters.

- $u'' + (u^2 - 1)u' + u^3 = 0.$
- $u'' + (u^4 - u^2)u' + u = 0.$
- $u'' + (u^2 - 1)u' + 2u + \sin u = 0.$
- $u'' + (u^2 - 1)u' + \arctan u = 0.$
- $u'' + (u^4 - 1)u' + \frac{u}{1+u^2} = 0.$