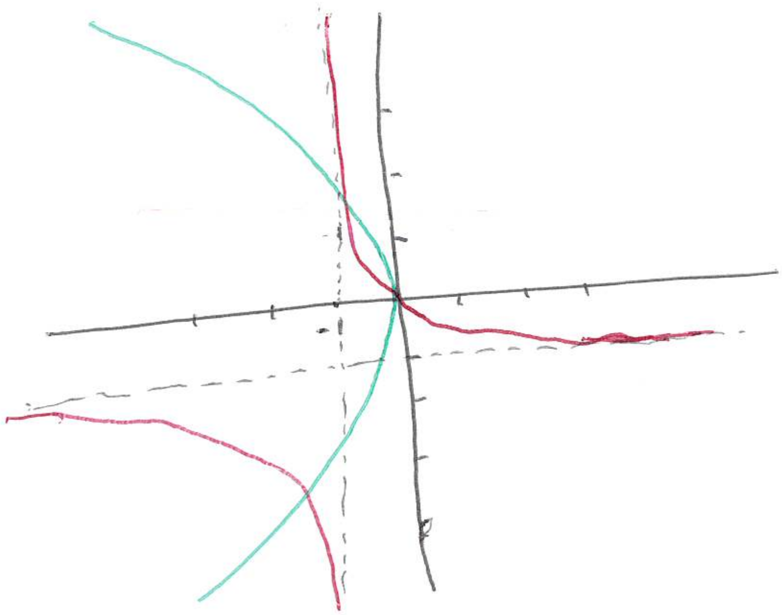
Example! We study the system $x' = 2x + y^2;$
$y' = x + y + xy.$

1. Find critical points.

$x' = 0 \iff x = -\frac{1}{2}y^2$ (green curve below)

$y' = 0 \iff y = \frac{-x}{x+1}$ (red curve below)



We can see that there should be 3 critical points.

But, the red and green curves give more info as we will see shortly.

We have $y = \dfrac{\frac{1}{2}y^2}{1-\frac{1}{2}y^2} \iff y - \frac{1}{2}y^3 = \frac{1}{2}y^2,$

$\iff y^3 + y^2 - 2y = 0,$

$\iff y(y+2)(y-1) = 0,$

$\iff y = -2, 0, 1.$

If $y=0$, $x = -\frac{1}{2}(0)^2 = 0$. Thus $(0,0)$ is a critical pt.

If $y=1$, $x = \frac{-1}{2}$. Thus $(-\frac{1}{2}, 1)$ " " " "

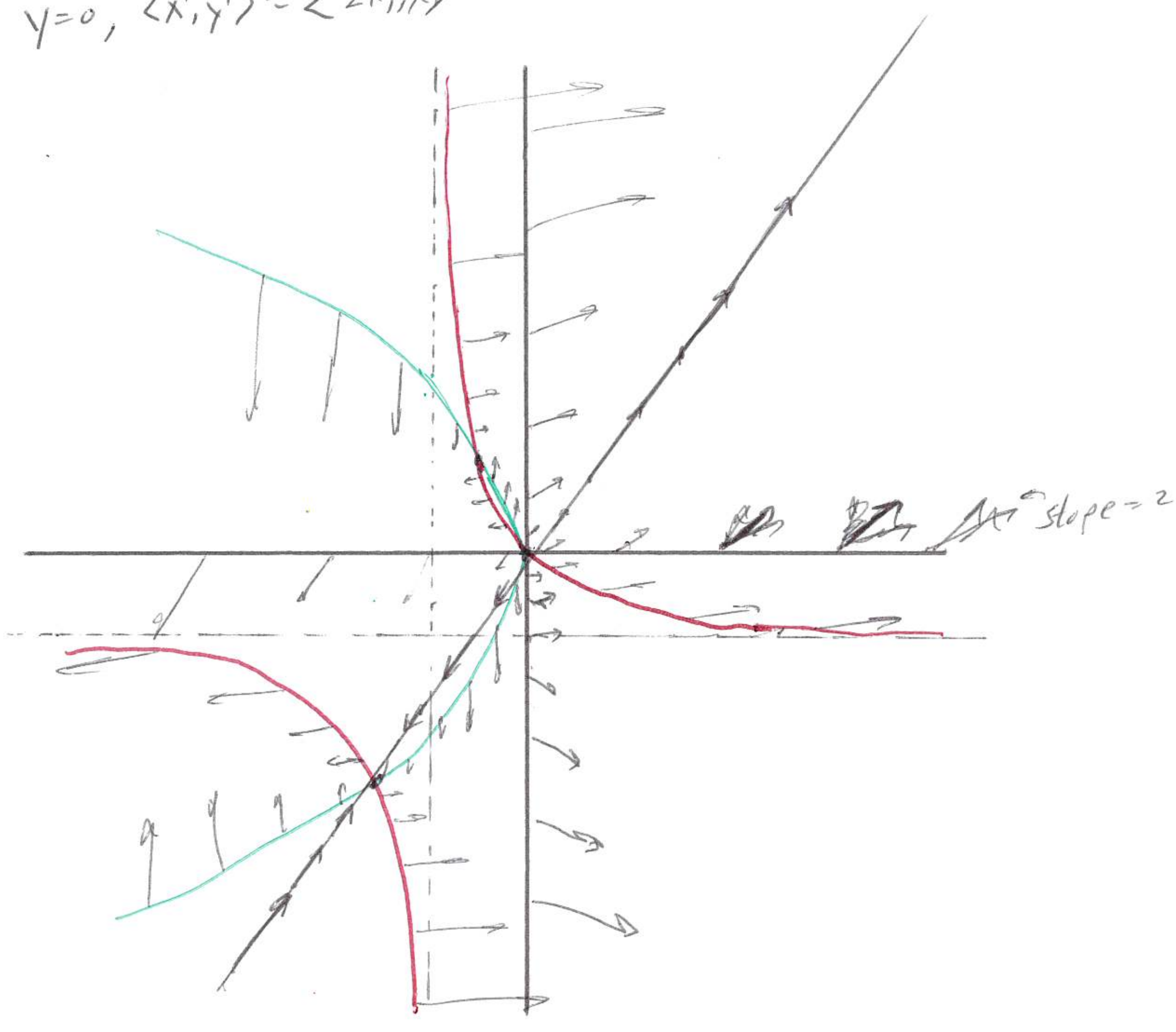
If $y=-2$, $x = -\frac{1}{2}(-2)^2 = -2$ Thus, $(-2,-2)$ " " " ".

2. Normally I'd find the linearizations at each cr. pt next. But, we are going spend some time just thinking about the vector field.

Along the green curve $x' = 0$, so the vectors, when not zero, are vertical. We can easily check if y' is + or −, and sketch in some vectors.

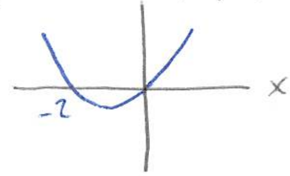
Along the red curve $y' = 0$, so the vectors, when not zero, are horizontal. It is easy to check if x' is + or −.

It is also easy to study $\langle x', y' \rangle$ along the x- and y-axes.
When $x=0$, $\langle x', y' \rangle = \langle y^2, y \rangle$.
When $y=0$, $\langle x', y' \rangle = \langle 2x, x \rangle$



Next, we notice something interesting. If $x=y$, $x' = 2x + x^2 = y'$. So, if we start on the line $x=y$, we will never leave it. Also, ~~slope~~ direction of $\langle x', y' \rangle$ is up/right for $x > 0$, or $x < -2$, and down/left for $-2 < x < 0$.

$x' = y' = 2x + x^2$. So, we can sketch in vectors along $x=y$ above



3. We can find explicit solution along the line $y = x$. This is rare.

$$\frac{dx}{dt} = 2x + x^2 \qquad \int \frac{1}{2x+x^2}\,dx = \int dt = t + c$$

$$\frac{1}{2x+x^2} = \frac{\frac{1}{2}}{x} - \frac{\frac{1}{2}}{x+2}. \qquad \frac{1}{2}\int \frac{1}{x} - \frac{1}{x+2}\,dx = \frac{1}{2}\ln|x| - \ln|x+2| + C$$

Thus,

$$\ln\left|\frac{x}{x+2}\right| = 2t + c$$

$$\left|\frac{x}{x+2}\right| = e^{2t+C} = Ce^{2t}$$

$$\frac{x}{x+2} = \pm C e^{2t} = Ce^{2t}.$$

We change C each step.

Thus $x(t) = \dfrac{2Ce^{2t}}{1 - Ce^{2t}} = \dfrac{2C}{e^{-2t} - C}$. $\quad y(t) = \dfrac{2C}{e^{-2t} - C}$ also.

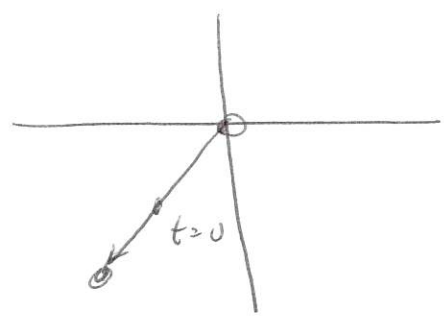
Initial values.

$(-1,-1)$ $\quad x(0) = -1.$ $\quad -1 = \dfrac{2C}{1-C} \Rightarrow -1 + C = 2C \Rightarrow C = -1.$

$$x(t) = \frac{-2}{e^{-2t}+1}. \qquad \text{Check. } x(0) = \frac{-2}{2} = -1 \checkmark.$$

$$\lim_{t\to\infty} x(t) = \frac{-2}{0+1} = -2. \qquad \lim_{t\to-\infty} x(t) = \frac{-2}{\text{"}\infty\text{"}+1} = 0. \text{ Same for } y(t).$$

Solution curve is



(1,1) This will get tricky!

$X(0)=1.$ $\quad 1=\frac{2c}{1-c} \Rightarrow c=\frac{1}{3}.$ So $x(t)=y(t)=\frac{2/3}{e^{-2t}-1/3}=\frac{2}{3e^{-2t}-1}.$

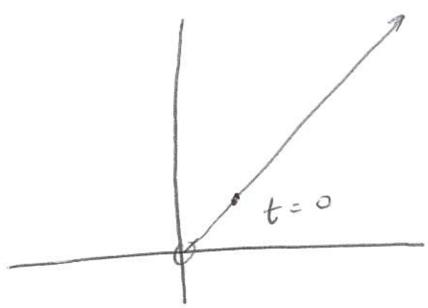
$X(0)=\frac{2}{3-1}=2$ ✓. $\quad \lim_{t\to-\infty} X(t)=\frac{2}{\text{"}\infty\text{"}-1}=0.$

But $\lim_{t\to\infty} x(t)=\frac{2}{0-1}=-2.$ Yikes!

Notice $3e^{-2t}-1$ will be zero when $t=\ln\sqrt{3}\approx 0.549.$

Thus $\lim_{t\to(\ln\sqrt{3})^-} X(t)=\infty.$

Same for $y(t)$. Thus, our solution curve is



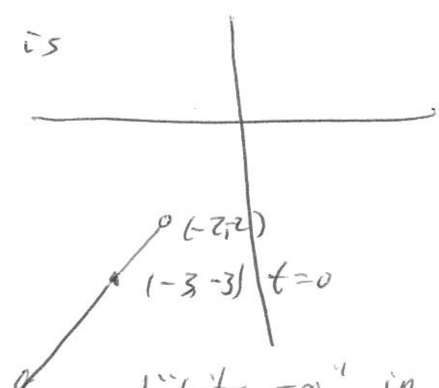
and "hits ∞" in finite time.

(-3,-3) $\quad X(0)=-3.$ Get $C=3.$ $\quad X(t)=Y(t)=\frac{6}{e^{-2t}-3}.$

$X(0)=\frac{6}{1-3}=-3.$ ✓ $\quad \lim_{t\to\infty} X(t)=\frac{6}{0-3}=-2.$ ✓

$\lim_{t\to-\infty} X(t)=0$ makes no sense. $\lim\, t\to-\ln\sqrt{3}$ from above

is $-\infty$. Solution curve is



and "hits $-\infty$" in finite time.

4. Finally we'll do the linearization.

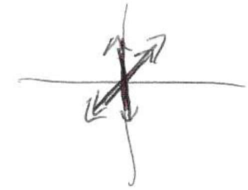
From the graph of the vector field it seems like $(0,0)$ is a repeller and the other two cr. pts, $(-2,-2)$ and $(-\frac{1}{2},1)$ will be saddles.

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 2x+y^2 \\ x+y+xy \end{bmatrix}. \quad \text{Thus } J = \begin{bmatrix} 2 & 2y \\ 1+y & 1+x \end{bmatrix}.$$

$$J(0,0) = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \qquad \det \begin{bmatrix} 2-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix} = (2-\lambda)(1-\lambda) \qquad \lambda = 1, 2$$ are e.values. repeller.

e.vectors: $\lambda = 1$ $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an e. vector.

$\lambda = 2$ $\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an e. vector



$$J(-2,-2) = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \qquad \begin{vmatrix} 2-\lambda & -4 \\ -1 & -1-\lambda \end{vmatrix} = (\lambda-2)(\lambda+1)-4$$

$$\lambda^2 - \lambda - 6$$

$$(\lambda-3)(\lambda+2)$$

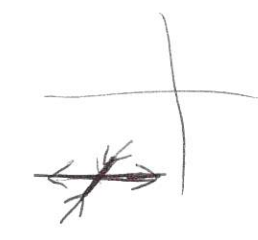
$\lambda = 3, -2$ saddle.

e.vectors: $\lambda = 3$

$\begin{bmatrix} -1 & -4 \\ -1 & -4 \end{bmatrix}$ $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$ is an e. vector.

$\lambda = -2$

$\begin{bmatrix} 4 & -4 \\ -1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an e. vector



$$J(-\tfrac{1}{2},1) = \begin{bmatrix} 2 & 2 \\ 2 & \frac{1}{2} \end{bmatrix} \qquad \begin{vmatrix} 2-\lambda & 2 \\ 2 & \frac{1}{2}-\lambda \end{vmatrix} = 1 - \tfrac{5}{2}\lambda + \lambda^2 - 4$$

$$= \lambda^2 - \tfrac{5}{2}\lambda - 3$$

$\lambda = \frac{5}{4} \pm \frac{1}{4}\sqrt{73}$ saddle

e. vector $\lambda = \frac{5+\sqrt{73}}{4}$

$\begin{bmatrix} \frac{-3-\sqrt{73}}{4} & 2 \end{bmatrix}$ e. vector is $\begin{bmatrix} 8 \\ -3-\sqrt{73} \end{bmatrix}$ other is is $\begin{bmatrix} 8 \\ -3+\sqrt{73} \end{bmatrix}$