Determinants[1]

1 Executive Summary

We present here just the bare bones of what students need to know about determinants of square matrices in order to apply and compute them.

1.1 Definitions and computations

We will define the **determinant** of a square ($n \times n$) matrix. Two notations are commonly used for the determinate of a square matrix A:

$$\det(A) \qquad\qquad \text{and} \qquad\qquad |A|$$

Don't confuse $|A|$ with the absolute value. Determinants can be negative.

Definition 1.1. For 2×2 matrices we define $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to be $ad - bc$.

If A is an $n \times n$ matrix, let A_{ij} be the $(n-1) \times (n-1)$ matrix formed by deleting the i^{th} row and j^{th} column of A. Assume the determinant of $(n-1) \times (n-1)$ matrices has been defined already. Then define the determinant of A by,

$$|A| = a_{11}|A_{11}| - a_{21}|A_{21}| + a_{31}|A_{31}| - \cdots \pm a_{n1}|A_{n1}|$$

$$= \sum_{i=1}^{n} (-1)^{i+1} a_{i1}|A_{i1}|.$$

This is called **expansion along the first column.**

Example 1. $\begin{vmatrix} 2 & 3 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 3 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} - \begin{vmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ 1 & 3 & -1 \end{vmatrix} +$

$$0\begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix} = 2 \times 3 - 9 + 0 + 4 = 1.$$

---

[1] ©Michael C. Sullivan, October 21, 2010

Problem 1. Find the determinants for the two matrices below by expanding along the first column.

$$A = \begin{bmatrix} 4 & -4 & 2 & 1 \\ 1 & 2 & 0 & 3 \\ 2 & 0 & 3 & 4 \\ 0 & -3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & -1 & 3 \\ 0 & 1 & 2 & 1 \\ 2 & -2 & 5 & 2 \\ 3 & 3 & 0 & 0 \end{bmatrix}$$

Answers: $\det(A) = 75$, $\det(B) = -135$.

It can be shown that the determinant can be computed just as well by expanding along any row or column where we adjust the signs as follows.

Theorem 1.2. *Let A be an $n \times n$ matrix. Then*

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{i1} |A_{ij}|$$

and

$$\det(A) = \sum_{i=j}^{n} (-1)^{i+j} a_{i1} |A_{ij}|.$$

In the first sum j is a fixed number between 1 and n that determines a fixed column of A. In the second sum i is a fixed number between 1 and n that determines a fixed row of A.

1.2 Effects of row & column operations

1. If A' is derived from A by multiplying a row or column by a real number r, then $\det(A') = r\det(A)$.
2. If A' is derived from A by adding a multiple of one row to another row or a multiple of one column to another column, then $\det(A') = \det(A)$
3. If A' is derived from A by switching two rows or by switching two columns, then $\det(A') = -\det(A)$.

An efficient method for computing determinants is to use row operations to convert A into an upper triangular matrix (one with all 0's below the diagonal). Then the determinant is just the product of the diagonal elements.

1.3 Important properties

1. $\det(AB) = \det(A)\det(B)$.
2. $\det(A^T) = \det(A)$.
3. A has an inverse if and only if $\det(A) \neq 0$.
4. If A^{-1} exists then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.
5. The rows and columns of A are linearly independent if and only if $\det A \neq 0$.

2 Introduction

We will define the **determinant** of a square $(n \times n)$ matrix. Two notations are commonly used for the determinate of a square matrix A:

$$\det(A) \qquad \text{and} \qquad |A|$$

It turns out that it is more natural to think of the determinant as a function whose input is n vectors from $\mathbb{R}^n$ – *e.g.* the row vectors of an $n \times n$ matrix – that outputs a real number. The essential nature of the determinant function is that it gives a way, and in some sense the only way, to define volume in spaces with dimension greater than three.

By using certain properties of the determinant we will develop some short cuts for computing determinants. These properties will also lead to important applications of the determinant function.

This section will move between three different levels of abstraction:

- computation of determinants,
- properties of the determinant function, and
- the essential nature of the determinant function as a volume function.

Pay careful attention to the interaction between these levels.

It is advisable to skip the proofs on the first reading. Some instructors may wish to skip some of the harder proofs altogether. If fact, there are several theorems in this chapter whose proofs are not give at all or are just outlined. These proofs are usually covered in more advanced courses.

3 Determinants for n = 1, 2 and 3

Let $n = 1$. Then we define $\det([a])$ to be a. Geometrically the is the length of the vector $\langle a \rangle$ in $\mathbb{R}^1$, except that it can be negative: $\det([-3]) = -3$. The sign of $\det([a])$ tells us the direction the vector points. We say $\det([a])$ is the *signed length* or the *oriented length* of the vector.

Let $n = 2$. We define $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to be $ad - bc$. It can be shown that $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the ± the area of the parallelogram determined by the row vectors, $[a\,b]$ and $[c\,d]$. Here are three properties of the determinant of 2×2 matrices. Their proofs are left to the you. Each has a geometric interpretation.

1. If A' is obtained from A by switching the rows, then $\det(A') = -\det(A)$.
2. If A' is obtained from A by multiplying either row by a real number c, then $\det(A') = c\det(A)$. Notice that this means $\det(cA) = c^2 \det(A)$.
3. If A' is obtained from A by adding a multiple of one row to the other, then $\det(A') = \det(A)$. This may seem surprising at first.

Problem 1. Prove the three properties of 2×2 determinants listed above.

Problem 2. Prove that $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the ± the area of the parallelogram determined by the row vectors, $[a\,b]$ and $[c\,d]$.

Property 1 says that taking the mirror image of a parallelogram changes its "orientation". Property 2 says that stretching a parallelogram in the direction of one edge by a factor of c, changes the area by a factor of c. Notice that if we stretch both edges by c the area changes by a factor of c^2.

Now for Property 3. What is it trying to tell us? The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ determines the rectangle $[1\,0] \times [0\,2]$, which has area $\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) = 2$. Now add three times the first row to the second to get $\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$. See Figure 1. Notice the new parallelogram still has base one and height two. Thus the areas are equal, as are the determinants! Property 3 says, loosely, that sliding a parallelogram along the direction of one of its edges does not change the area.

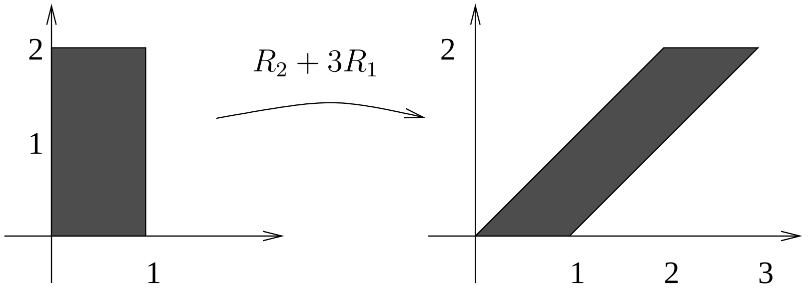

Figure 1: Area is unchanged.

It is easy to show that $\det(AB) = \det(A)\det(B)$, A is invertible if and only if $\det(A) \neq 0$, and $\det(A^T) = \det(A)$. Later in this section we will show that these statements hold for all $n \times n$ matrices.

On to $n = 3$. We define the determinant of a 3×3 matrix by the formula below.

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - d \begin{vmatrix} b & b \\ h & i \end{vmatrix} + g \begin{vmatrix} b & c \\ e & f \end{vmatrix}.$$

This is called **expansion along the first column.**

Problem 3. Compute the following determinants.

$$\text{a. } \begin{vmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{vmatrix} \qquad \text{b. } \begin{vmatrix} 2 & 3 & -2 \\ 0 & 5 & 8 \\ 0 & 0 & 7 \end{vmatrix} \qquad \text{c. } \begin{vmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \\ -1 & 3 & 0 \end{vmatrix}.$$

The determinant of a 3×3 matrix does give $\pm$ the volume of the parallelepiped determined by its row vectors. Let's look at an easy example. Consider 3×3 diagonal matrices. In this case $\det \begin{bmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{bmatrix} = aei$ which is the signed volume of the rectangular prism $[a, 0, 0] \times [0, e, 0] \times [0, 0, i]$.

Problem 4. Prove that the determinant of a 3×3 matrix is $\pm$ the volume of the parallelepiped determined by its row vectors. (This can be found in most third semester calculus textbooks.)

The effects of performing row operations on 3×3 matrices is the same as for 2×2 matrices.

Problem 5. Prove the following statements for 3×3 matrices.
a. If A' is obtained from A by switching any two rows, then $\det(A') = -\det(A)$.
b. If A' is obtained from A by multiplying any row by a real number c, then $\det(A') = c\det(A)$. Notice that this means $\det(cA) = c^3\det(A)$.
c. If A' is obtained from A by adding a multiple of one row to another, then $\det(A') = \det(A)$. (This is hard and is done in the next section.)
Isn't this just what a volume function should do?

Problem 6. Let A be a 3×3 matrix.
a. What can you say about the determinant if two rows are the same?
b. What can you say about the determinant if A has a row of zeros?
c. Prove $\det(A) = \det(A^T)$. (This shows that the parallelepiped determined by the rows of A has the same oriented volume as the one determined by the columns of A.)
d. What can you say about the determinant if two column are the same? What can you say about the determinant if A has a column of zeros?
e. Prove that if B is also a 3×3 matrix then $\det(AB) = \det(A)\det(B)$.
f. Prove that A is invertible (nonsingular) if and only if $\det(A) \neq 0$.
g. Prove that if A is invertible then $\det(A^{-1}) = 1/\det(A)$. Hint: what is $\det(I)$?

Example 1. Find $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{vmatrix}$. *Solution.* Subtract row 1 from row two. This does not change the value of the determinant. Since we now have two identical rows, the determinant must be zero.

4 Recursive Definition of the Determinant

Now we define, recursively, the determinant of an $n \times n$ matrix.

Definition 4.1. If A is an $n \times n$ matrix, let A_{ij} be the $(n-1) \times (n-1)$ matrix formed by deleting the i^{th} row and j^{th} column of A. Then define the determinant of A by,

$$|A| = a_{11}|A_{11}| - a_{21}|A_{21}| + a_{31}|A_{31}| - \cdots \pm a_{n1}|A_{n1}|$$

$$= \sum_{i=1}^{n} (-1)^{i+1} a_{i1} |A_{i1}|.$$

This again is called *expansion along the first column.*

Example 1. $\begin{vmatrix} 2 & 3 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 3 & -1 \end{vmatrix} = 2 \begin{vmatrix} 1 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} - \begin{vmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ 1 & 3 & -1 \end{vmatrix} +$

$$0 \begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix} - (-1) \begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix} = 2 \times 3 - 9 + 0 + 4 = 1.$$

Problem 1. Find the determinants for the two matrices below by expanding along the first column.

$$A = \begin{bmatrix} 4 & -4 & 2 & 1 \\ 1 & 2 & 0 & 3 \\ 2 & 0 & 3 & 4 \\ 0 & -3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & -1 & 3 \\ 0 & 1 & 2 & 1 \\ 2 & -2 & 5 & 2 \\ 3 & 3 & 0 & 0 \end{bmatrix}$$

Answers: $\det(A) = 75$, $\det(B) = -135$.

Next we establish two lemmas – a lemma is a small theorem that is used to prove a larger theorem. These two lemmas will be used in the next section. The first one is interesting is its own right.

Lemma 4.2. *Let A be an $n \times n$ matrix and let A' be obtained from A by switching any two rows. Then* $\det(A') = -\det(A)$*.*

Partial Proof. [2]

Step 1. First we consider a special case. Suppose A' is obtained from A by switching two adjacent rows. We will show $\det(A') = -\det(A)$. We will assume that this is true for 3×3 matrices (Problem 5a), and just work out the 4×4 case. To simplify notation we will switch the second and third rows of a 4×4 matrix. Other adjacent row switchings are done in the same way.

Let

$$A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} R_1 \\ R_3 \\ R_2 \\ R_4 \end{bmatrix}.$$

---

[2] Based on Len Evans' notes.

Now

$$|A'| = a'_{11}|A'_{11}| - a'_{21}|A'_{21}| + a'_{31}|A'_{31}| - a'_{41}|A'_{41}|. \qquad (\#)$$

Notice that

$$a'_{11} = a_{11} \text{ and } a'_{41} = a_{41} \text{ while } a'_{21} = a_{31} \text{ and } a'_{31} = a_{21}.$$

That is, $a'_{j1} = a_{j1}$, unless j corresponds to one of the rows being switched, in which case they switch with the rows. Also notice that if j is not one of the row being switched then A'_{j1} can be obtained from A_{j1} by switching two adjacent rows. Since these are 3×3 we have

$$|A'_{11}| = -|A_{11}| \text{ and } |A'_{41}| = -|A_{41}|$$

What happens to $|A'_{21}|$ and $|A'_{31}|$? We can check that $A'_{21} = A_{31}$ and $A'_{31} = A_{21}$. (This is where the adjacency assumption is needed.) It follows that

$$|A'_{21}| = |A_{31}| \text{ and } |A'_{31}| = |A_{21}|$$

Next we apply what we have learned to equation (#) to get

$$|A'| = -a_{11}|A_{11}| - a_{31}|A_{31}| + a_{21}|A'_{21}| + a_{41}|A_{41}| =$$

$$-1(a_{11}|A_{11}| - a_{21}|A_{21}| + a_{31}|A'_{31}| - a_{41}|A_{41}|) = -|A|.$$

The $n \times n$ case can be done by induction.

Step 2. Now we drop the assumption that the rows to be switched are adjacent. Suppose A is an $n \times n$ matrix and A' is obtained from A by switching rows j and k, with $j < k$. In Problem 2 You will show that switching any two rows can be decomposed into an odd number of adjacent row switchings. But, -1 raised to an odd power is -1. Thus, $|A'| = -|A|$. □

Problem 2. Consider a list of n symbols where $n \geq 2$. We wish to switch the first symbol with last symbol leaving the others unchanged. But, we can only switch two adjacent symbols at a time; call this operation an *adjacent switch move.* Prove that the number of adjacent switch moves needed to switch the first and last symbol, leaving the others unchanged, is always an odd number. We will use this result later.

Lemma 4.3. *Let A, A' and A'' be $n \times n$ matrices that differ only in their k-th rows and suppose that the k-th row of A is the sum of the k-th rows of A' and A''. Then*

$$\det(A) = \det(A') + \det(A'').$$

Proof of 3×3 *case.* The lemma is easy to check for the 2×2 case. Let,

$$A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix},\ A' = \begin{bmatrix} R_1 \\ R_2 \\ R_3' \end{bmatrix} \text{ and } A'' = \begin{bmatrix} R_1 \\ R_2 \\ R_3'' \end{bmatrix},$$

where $R_3 = R_3' + R_3''$. The proofs for splitting rows 1 or 2 are similar. By expanding along the first column we get

$$|A| = a_{11}|A_{11}| - a_{21}|A_{21}| + a_{31}|A_{31}| \qquad (*)$$

and

$$|A'| + |A''| = a_{11}'|A_{11}'| - a_{21}'|A_{21}'| + a_{31}'|A_{31}'| + a_{11}''|A_{11}''| - a_{21}''|A_{21}''| + a_{31}''|A_{31}''|. \,(**)$$

We want to show these two equations are equal. Of course we have

$$a_{11} = a_{11}' = a_{11}'',\ \ a_{21} = a_{21}' = a_{21}'' \text{ and } a_{31} = a_{31}' + a_{31}''.$$

But also, since the lemma holds for 2×2 matrices,

$$|A_{11}| = |A_{11}'| + |A_{11}''| \text{ and } |A_{21}| = |A_{21}'| + |A_{21}''|.$$

Finally, when we delete the third row we get

$$|A_{31}| = |A_{31}'| = |A_{31}''|,$$

because $A_{31} = A_{31}' = A_{31}''$.

We apply these to equation $(*)$.

$$|A| = a_{11}(|A_{11}'| + |A_{11}''|) - a_{21}(|A_{21}'| + |A_{21}''|) + (a_{31}' + a_{31}'')|A_{31}| =$$

$$a_{11}|A_{11}'| + a_{11}|A_{11}''| - a_{21}|A_{21}'| - a_{21}|A_{21}''| + a_{31}'|A_{31}| + a_{31}''|A_{31}| =$$
$$a_{11}'|A_{11}'| + a_{11}''|A_{11}''| - a_{21}'|A_{21}'| - a_{21}''|A_{21}''| + a_{31}'|A_{31}'| + a_{31}''|A_{31}''|.$$

By rearranging these terms we can use equation $(**)$ to get $|A| = |A'| + |A''|$. The general proof uses induction. □

5 The Determinant as a Volume Function

We shall say the a function from $M^{n\times n}$ (or n vectors from R^n) to the reals is an oriented volume function if it satisfies the these three properties.

1. $\mathrm{vol}(I) = 1$; that is, the unit cube should have volume one in all dimensions.
2. If A' is obtained from A by multiplying a row of A by a real number c, then $\mathrm{vol}(A') = c\mathrm{vol}(A)$ – the stretching property.
3. If A' is obtained from A by adding one row of A to another, then $\mathrm{vol}(A') = \mathrm{vol}(A)$ – the sliding property.

Not only does the determinant function satisfy these three properties, it is the only function to do so!

Theorem 5.1 (The Fundamental Theorem of Determinants)**.** *The determinate is the unique function that satisfies the three properties above.*

First we shall give a partial proof that the determinant function satisfies the three properties. Then we do the uniqueness proof after making a remark about multi-linear functions.

Partial Proof of Properties. 1. The first property should be obvious, but we could do it by induction on n where I_n is the $n \times n$ identity matrix.

2. You should have worked out the 2×2 and 3×3 cases (Problems 1 and 5b respectively). We will do the 4×4 case in such a way that you should be able to see how to get the general case by induction. Let $A = [a_{ij}]$ be a 4×4 matrix. Multiply the first row by c to obtain A'.

Now,

$$|A'| = ca_{11}|A'_{11}| - a_{21}|A'_{21}| + a_{31}|A'_{31}| - a_{41}|A'_{41}|.$$

Since A'_{11} is the same as A_{11} (A' only differs from A in the first row), we have $|A'_{11}| = |A_{11}|$.

Next notice that A'_{21} can be obtained from A_{21} by multiplying the first row of A_{21} by c. Thus, $|A'_{21}| = c|A_{21}|$, since we know Property 2 holds for 3×3 matrices. Likewise, $|A'_{31}| = c|A_{31}|$ and $|A'_{41}| = c|A_{41}|$. We can now write

$$|A'| = ca_{11}|A_{11}| - a_{21}c|A_{21}| + a_{31}c|A_{31}| - a_{41}c|A_{41}|.$$

If we factor out the c then we get $|A'| = c|A|$. The argument is similar for any row of A. This establishes Property 2 for 4×4 matrices. Hopefully you can see how one could use the Principle of Mathematical Induction to get the general result.

3. Property 3 is the hardest to prove. Here is where we use Lemmas 4.2 and 4.3. Property 3 is easy to verify for 2×2 matrices (Problem 1). We assume this and do the 3×3 case. (The 3×3 case was done in Problem 5c. But the proof here is set up so that you can see how it might be generalized by induction for $n \times n$ matrices.) Let

$$A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \text{ and } A' = \begin{bmatrix} R_1 \\ R_2 \\ R_3 + R_1 \end{bmatrix}$$

Other row additions are done similarly. Now then,

$$|A| = a_{11}|A_{11}| - a_{21}|A_{21}| + a_{31}|A_{31}| \qquad (!)$$

and

$$|A'| = a'_{11}|A'_{11}| - a'_{21}|A'_{21}| + a'_{31}|A'_{31}|. \qquad (!!)$$

We will transform (!!) into (!). We have

$$a'_{11} = a_{11},\ \ a'_{21} = a_{21} \text{ and } a'_{31} = a_{31} + a_{11}.$$

Also, $A'_{31} = A_{31}$ implies $|A'_{31}| = |A_{31}|$ and $|A'_{21}| = |A_{21}|$ follows from the 2×2 case. Now,

$$A'_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} + a_{12} & a_{33} + a_{13} \end{bmatrix}.$$

By Lemma 4.3

$$|A'_{11}| = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{12} & a_{13} \end{vmatrix} =$$

$$|A_{11}| - |A_{31}|.$$

The fact that $\begin{vmatrix} a_{22} & a_{23} \\ a_{12} & a_{13} \end{vmatrix} = -|A_{31}|$ comes from Lemma 4.2 on row switching. We apply what we know to equation (!!) to get

$$|A'| = a_{11}(|A_{11}| - |A_{31}|) - a_{21}|A_{21}| + (a_{31} + a_{11})|A_{31}| =$$

$$a_{11}|A_{11}| - a_{21}|A_{21}| + a_{31}|A_{31}| = |A|.$$

This completes our outline of the proof that the determinant function satisfies properties 1-3. $\square$

Problem 1. The partial proof just given was set up in such a way that the reader should be able to see how the general proof would work. The only way to know if this has been achieved is for the reader to work out the full proof.
a. Use the Principle of Mathematical Induction to proof that $\det I = 1$ for all $n \times n$ identity matrices.
b. Give a full proof for Property 2. Yo u might start by writing out the proof for $n = 5$ or 6 in your own words and then doing the most general case.
c. Do the proof of Property 3 for $n = 4$ switching rows 2 and 3.
d. Do the proof of Property 3 for an arbitrary n switching and p and q.

Remark. Let $n \geq 1$ and $1 \leq p \leq n$. Let R_1,..., R_{p-1}, R_{p+1},...,R_n be in $\mathbb{R}^n$. Define

$$T : \mathbb{R}^n \to \mathbb{R}$$

by

$$T(\mathbf{v}) = \det \begin{bmatrix} R_1 \\ \vdots \\ R_{p-1} \\ \mathbf{v} \\ R_{p+1} \\ \vdots \\ R_n \end{bmatrix}.$$

Then T is linear. This follows from Property 2 and Lemma 4.3.

Definition 5.2. A function $M : (V)^m \to W$, where V and W are vector spaces, is **multi-linear** if

$$M(\mathbf{v}_1, ..., \mathbf{v}_{p-1}, c\mathbf{v}_p, \mathbf{v}_{p+1}, ..., \mathbf{v}_m) = cM(\mathbf{v}_1, ..., \mathbf{v}_m)$$

and

$$M(\mathbf{v}_1, ..., \mathbf{v}_{p-1}, \mathbf{v}'_p + \mathbf{v}''_p, \mathbf{v}_{p+1}, ..., \mathbf{v}_m) =$$
$$M(\mathbf{v}_1, ..., \mathbf{v}_{p-1}, \mathbf{v}'_p, \mathbf{v}_{p+1}, ..., \mathbf{v}_m) + M(\mathbf{v}_1, ..., \mathbf{v}_{p-1}, \mathbf{v}''_p, \mathbf{v}_{p+1}, ..., \mathbf{v}_m).$$

The determinant function, indeed any function from $(\mathbb{R}^n)^n$ (or $M^{n\times n}$) to $\mathbb{R}$ that satisfies Property 2 and Lemma 4.3, is multi-linear. We also know that if we switch two rows the determinant function changes sign. Such functions said to be **alternating**. The determinant then is an **alternating multi-linear** function.

Now we move on to the uniqueness part of the proof of Theorem 5.1.

Proof of Uniqueness. **Step 1.**Suppose the function $\mathrm{vol} : M^{n\times n} \to \mathbb{R}$ satisfies Properties 1-3. We will show that if two rows are switched then vol changes sign. It follows that if two rows are equal then vol is zero. Each equality below follows from Property 2 or 3; the reader should determine which Property is use for each step.

$$\mathrm{vol}\begin{bmatrix} \vdots \\ R_p \\ \vdots \\ R_q \\ \vdots \end{bmatrix} = -\mathrm{vol}\begin{bmatrix} \vdots \\ -R_p \\ \vdots \\ R_q \\ \vdots \end{bmatrix} = -\mathrm{vol}\begin{bmatrix} \vdots \\ -R_p \\ \vdots \\ R_q - R_p \\ \vdots \end{bmatrix} = \mathrm{vol}\begin{bmatrix} \vdots \\ -R_p \\ \vdots \\ R_p - R_q \\ \vdots \end{bmatrix} = \mathrm{vol}\begin{bmatrix} \vdots \\ -R_q \\ \vdots \\ R_p - R_q \\ \vdots \end{bmatrix} =$$

$$-\mathrm{vol}\begin{bmatrix} \vdots \\ R_q \\ \vdots \\ R_p - R_q \\ \vdots \end{bmatrix} = -\mathrm{vol}\begin{bmatrix} \vdots \\ R_q \\ \vdots \\ R_p \\ \vdots \end{bmatrix}$$

Step 2. We will show that vol satisfies the conclusion of Lemma 4.3. This step has four substeps.

Substep 2a. If one of the vectors is the zero vector then vol is zero since by Property 2 we have

$$\mathrm{vol}\begin{bmatrix} \vdots \\ \mathbf{0} \\ \vdots \end{bmatrix} = \mathrm{vol}\begin{bmatrix} \vdots \\ 0\mathbf{0} \\ \vdots \end{bmatrix} = 0\mathrm{vol}\begin{bmatrix} \vdots \\ \mathbf{0} \\ \vdots \end{bmatrix} = 0.$$

Substep 2b. If A' is obtained from A by adding a multiply of one row to a different row then $\mathrm{vol}A' = \mathrm{vol}A$. Here is the proof. Let $c \neq 0$.

$$\mathrm{vol}\begin{bmatrix} \vdots \\ R_P \\ \vdots \\ R_q \\ \vdots \end{bmatrix} = \frac{1}{c}\mathrm{vol}\begin{bmatrix} \vdots \\ cR_P \\ \vdots \\ R_q \\ \vdots \end{bmatrix} = \frac{1}{c}\mathrm{vol}\begin{bmatrix} \vdots \\ cR_P \\ \vdots \\ R_q + cR_P \\ \vdots \end{bmatrix} = \frac{c}{c}\mathrm{vol}\begin{bmatrix} \vdots \\ R_P \\ \vdots \\ R_q + cR_P \\ \vdots \end{bmatrix} = \mathrm{vol}\begin{bmatrix} \vdots \\ R_P \\ \vdots \\ R_q + cR_P \\ \vdots \end{bmatrix}.$$

Substep 2c. If $\{R_1, ..., R_n\}$ is linearly dependent then vol $\begin{bmatrix} R_1, ..., R_n \end{bmatrix} = 0$. Some R_j is a linear combination of the remaining rows. Suppose $j = 1$ and that $R_1 = \sum_{j=2}^n a_j R_j$. The argument for the other rows is essentially the same. By repeatedly applying Step 2b we have

$$\mathrm{vol}\begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix} = \mathrm{vol}\begin{bmatrix} R_1 - \sum_{j=2}^n a_j R_j \\ R_2 \\ \vdots \\ R_n \end{bmatrix} = \mathrm{vol}\begin{bmatrix} \mathbf{0} \\ R_2 \\ \vdots \\ R_n \end{bmatrix} = 0.$$

Substep 2d. Let

$$A = \begin{bmatrix} R_1' + R_1'' \\ R_2 \\ \vdots \\ R_n \end{bmatrix}, \quad A' = \begin{bmatrix} R_1' \\ R_2 \\ \vdots \\ R_n \end{bmatrix}, \text{ and } A'' = \begin{bmatrix} R_1'' \\ R_2 \\ \vdots \\ R_n \end{bmatrix}.$$

We shall assume $\{R_2, ..., R_n\}$ is linearly independent since otherwise $\mathrm{vol}A = \mathrm{vol}A' = \mathrm{vol}A'' = 0$ and the result holds. Let $\{\mathbf{v}, R_2, ..., \mathbb{R}_n\}$ be a basis for $\mathbb{R}^n$ and let

$$R_1' = \alpha_1\mathbf{v} + \alpha_2 R_2 + \cdots + \alpha_n R_n, \text{ and } R_1'' = \beta_1\mathbf{v} + \beta_2 R_2 + \cdots + \beta_n R_n.$$

Then by repeatedly applying Step 2b we have

$$\mathrm{vol}A' = \mathrm{vol}\begin{bmatrix} \alpha_1\mathbf{v} + \alpha_2 R_2 + \cdots + \alpha_n R_n \\ R_2 \\ \vdots \\ R_n \end{bmatrix} = \mathrm{vol}\begin{bmatrix} \alpha_1\mathbf{v} \\ R_2 \\ \vdots \\ R_n \end{bmatrix} = \alpha_1\mathrm{vol}\begin{bmatrix} \mathbf{v} \\ R_2 \\ \vdots \\ R_n \end{bmatrix}.$$

Likewise, $\mathrm{vol}A'' = \beta_1 \mathrm{vol}\begin{bmatrix} \mathbf{v} \\ R_2 \\ \vdots \\ R_n \end{bmatrix}$ and $\mathrm{vol}A = (\alpha_1 + \beta_1)\mathrm{vol}\begin{bmatrix} \mathbf{v} \\ R_2 \\ \vdots \\ R_n \end{bmatrix}$. Thus,

$$\mathrm{vol}A = \alpha_1 \mathrm{vol}\begin{bmatrix} \mathbf{v} \\ R_2 \\ \vdots \\ R_n \end{bmatrix} + \beta_1 \mathrm{vol}\begin{bmatrix} \mathbf{v} \\ R_2 \\ \vdots \\ R_n \end{bmatrix} = \mathrm{vol}A' + \mathrm{vol}A''.$$

If follows that *vol* and det are both alternating multi-linear functions.

Step 3. Now let $\Delta(R_1, ..., R_n) = \det \begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix} - \mathrm{vol}\begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix}$. We will show that $\Delta(R_1, ..., R_n)$ is always zero. Clearly Δ is alternating and multi-linear. For each $i = 1, ..., n$ let $R_i = \sum_{j=1}^{n} c_{ij}\mathbf{e}_j$, where the $\mathbf{e}_j$ are the standard basis elements for $\mathbb{R}^n$. Then

$$\Delta(R_1, ..., R_n) = \Delta\left(\sum_{j=1}^{n} c_{1j}\mathbf{e}_j, R_2, ..., R_n\right) = \sum_{j=1}^{n} c_{1j}\Delta(\mathbf{e}_j, R_2 ..., R_n)$$

$$= \sum_{j_1=1}^{n} c_{1j_1}\Delta(\mathbf{e}_{j_1}, \sum_{j_2=1}^{n} c_{2j_2}\mathbf{e}_{j_2}, R_3, ..., R_n)$$

$$= \sum_{j_1=1}^{n}\sum_{j_2=1}^{n} c_{1j_1}c_{2j_2}\Delta(\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, R_3, ..., R_n) = \cdots$$

$$= \sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\cdots\sum_{j_n=1}^{n} c_{1j_1}c_{2j_2}\cdots c_{nj_n}\Delta(\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, ..., \mathbf{e}_{j_n})$$

If the $\mathbf{e}_{j_1}$, $\mathbf{e}_{j_2}$,...,$\mathbf{e}_{j_n}$ are distinct, then $\Delta(\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, ..., \mathbf{e}_{j_n}) = \pm\Delta(\mathbf{e}_1, ..., \mathbf{e}_n) = \pm(0 - 0) = 0$. If not then two of them are equal and $\Delta(\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, ..., \mathbf{e}_{j_n}) = 0$. Thus $\Delta(R_1, ..., R_n) = 0$. ☐

6 Determinants and Row Operations

The next theorem records the effect of the three row operations on the determinant of a matrix. These facts have already been established but it is helpful to view them in the context of row operations.

Theorem 6.1 (The Effect of Row Operations)**.** *1. If A' is obtained from A by switching two rows, then* $\det A' = -\det A$.

2. If A' is obtained from A by multiplying a row by a nonzero constant c, then $\det A' = c \det A$.

3. If A' is obtained from A by adding the multiple of one to another then $\det A' = \det A$.

Proof. Statement 1 is just Lemma 4.2. Statement 2 is Property 2 in Theorem 5.1. Statement 3 was shown in Step 2b of the proof of the uniqueness part of Theorem 5.1. ☐

Theorem 6.1 gives a tool for simplifying the computing of determinants. We use row operations to transform A into an upper triangular matrix, tracking how the row operations effect the determinant. It is easy to show that the determinant of an upper triangular matrix is just the product of its diagonal entries. From this information we can deduce the determinant of A.

Example 1 (Upper triangular matrices)**.** In the following example (from Len Evans', *A Brief Course in Linear Algebra*) identify the operation being used.

$$\begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 3 & 0 & 1 & 1 \\ -1 & 6 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & -6 & 4 & -2 \\ 0 & 8 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 7 & 4 \\ 0 & 0 & -5 & -5 \end{vmatrix} =$$

$$-5\begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 7 & 4 \\ 0 & 0 & 1 & 1 \end{vmatrix} = +5\begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 7 & 4 \end{vmatrix} = +5\begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{vmatrix}.$$

The last matrix is an upper triangular matrix. Its determinant is especially easy to compute.

$$\begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{vmatrix} = 1 \cdot 2 \cdot \begin{vmatrix} 1 & 1 \\ 0 & -3 \end{vmatrix} = 1 \cdot 2 \cdot 1 \cdot (-3) = -6.$$

Thus the determinant of the original matrix is $5 \cdot (-6) = -30$.

Problem 1. a. Prove that if a square matrix has two identical rows its determinant is zero.
b. Prove that if a square matrix has a row of zeros its determinant is zero.

7 Two Big Theorems

Theorem 7.1. A^{-1} *exists if and only if* $\det A \neq 0$.

Theorem 7.2. $\det AB = \det A \det B$.

Remark. Both of these can be made intuitively plausible by thinking in terms of volume.

Proof of Theorem 7.1. The square matrix A is nonsingular (i.e. invertible) if and only if there exists a sequence of row operations taking A to I.

If $\det A = 0$ any matrix derived from A by row operations will also have zero determinant. Hence A is not row equivalent to I and so A^{-1} does not exist.

Suppose now that A is known to be not invertible. Let $B = \operatorname{rref}(A)$. Then B cannot have a complete set of pivots, that is B must have a zero on its diagonal. But B is an upper triangular matrix (because it is reduced). Thus $\det B = 0$, which implies $\det A = 0$. □

Proof of Theorem 7.2. First we establish the following fact. Let A and B be matrices such that B can be derived from A by a single row operation which we denote by r, i.e.

$$A \xrightarrow{r} B.$$

Now let C be a third matrix and consider the products AC and BC. Our claim is that if you apply the same row operation r to AC you get BC,

$$AC \xrightarrow{r} BC.$$

Here we will show this for Row Operation 3 for 3 × 3 matrices. You should be able to see how this could be extended to cover $n \times n$ matices; induction is not needed. In Problem 1 you will show the equavalent result for the other two Row Operations.

Let

$$A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}, \qquad B = \begin{bmatrix} R_1 \\ R_2 \\ R_3 + cR_2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix}.$$

Then

$$AC = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix}$$

and

$$BC = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ (R_3 + kR_2)C_1 & (R_3 + kR_2)C_2 & (R_3 + cR_2)C_3 \end{bmatrix} =$$

$$\begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 + cR_2C_1 & R_3C_2 + cR_2C_2 & R_3C_3 + cR_2C_3 \end{bmatrix}.$$

Thus original row operation takes AC to BC. We are now in position to prove our theorem. The proof is divided into two cases.

Case 1: Suppose A is nonsingular ($\det A \neq 0$). Then there are k row operations taking A to I, for some number k:

$$A = A_0 \xrightarrow{r_1} A_1 \xrightarrow{r_2} A_2 \xrightarrow{r_3} \cdots \xrightarrow{r_k} A_k = I$$

For each r_i there is a nonzero number c_i (which could be 1) such that $\det A_{i-1} = c_i \det A_i$. Thus,

$$\det A = c_1 \det A_1 = c_1 c_2 \det A_2 = \cdots = c_1 c_2 c_3 \cdots c_k \det I,$$

and so we can write $\det A = c_1 c_2 \cdots c_k$. Now apply exactly the same row operations to the product AB.

$$AB \xrightarrow{r_1} A_1 B \xrightarrow{r_2} A_2 B \xrightarrow{r_3} \cdots \xrightarrow{r_k} IB.$$

Thus we have

$$\det AB = c_1 \cdots c_k \det B = \det A \det B.$$

Case 2: Suppose $\det A = 0$. We must show that $\det AB = 0$ since $\det A \det B = 0 \cdot \det B = 0$.

Since A is not invertible there is a sequence of row operations taking A to a matrix Z that has a row of zeros. (Why? A good test question!?)

$$A = A_0 \xrightarrow{r_1} A_1 \xrightarrow{r_2} A_2 \xrightarrow{r_3} \cdots \xrightarrow{r_k} A_k = Z.$$

Thus,

$$AB \xrightarrow{r_1} A_1B \xrightarrow{r_2} A_2B \xrightarrow{r_3} \cdots \xrightarrow{r_k} ZB.$$

Then

$$\det AB = c_1 \cdots c_k \det ZB.$$

But if Z has a row of zeros so does ZB (check this!). Thus, $\det ZB = 0$. This completes our proof. □

Problem 1. a. Suppose martix B is obtained from A by Row Operation 1 and let C be a matrix such that AC and BC are defined. Prove that BC can be obtained from AC by applying the same row operation. You may do just the 3×3 or th emore general case as you prefer.
b. Suppose martix B is obtained from A by Row Operation 2 and let C be a matrix such that AC and BC are defined. Prove that BC can be obtained from AC by applying the same row operation. You may do just the 3×3 or th emore general case as you prefer.

8 Transposes and another short cut

Theorem 8.1 (Transposes)**.** *$\det A^T = \det A$.*

Proof. The dimension of the row space and the column space of a matrix are equal. It follows that A is invertible if and only if A^T is invertible. If A is not invertible $\det A = 0$ and since A^T is not invertible $\det A^T = 0$.

Assume A and hence A^T are invertible. Then A can be factored into elementary matrices $A = E_1E_2 \cdots E_n$. Then $A^T = E_n^T E_{n-1}^T \cdots E_1^T$. It is easy to check that $\det E = \det E^T$ for any elementary matrix. Applying the determinate product rule repeatedly gives

$$\det A = \det E_1 \cdot \det E_2 \cdots \det E_n = \det E_1^T \cdot \det E_2^T \cdots \det E_n^T = \det A^T.$$

□

We can now produce more tools for calculating determinants.

Theorem 8.2. *1. Statements (1-3) in Theorem 6.1 remain true if "rows" is replaced with "columns".*
2. We can compute a determinant by expanding along any row or column, provided we watch our signs.

Outline of proof. Statement 1 follows from Theorem 8.1 and Theorem 6.1. For Statement 2 just use row switching and transposes to place the row or column you wish to expand along in the first column. Of course you have to watch how any row switching effects the signs. ☐

Example 1. We show that the determinant of a 3×3 matrix can be obtained by expand along the second row.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ h & i & j \end{vmatrix} = - \begin{vmatrix} d & e & f \\ a & b & c \\ h & i & j \end{vmatrix} = - \begin{vmatrix} d & a & g \\ e & b & h \\ f & c & i \end{vmatrix} =$$

$$- \left(d \begin{vmatrix} b & h \\ c & i \end{vmatrix} - e \begin{vmatrix} a & g \\ c & i \end{vmatrix} + f \begin{vmatrix} a & g \\ b & h \end{vmatrix} \right) =$$

$$-d \begin{vmatrix} b & h \\ c & i \end{vmatrix} + e \begin{vmatrix} a & g \\ c & i \end{vmatrix} - f \begin{vmatrix} a & g \\ b & h \end{vmatrix}.$$

This is the same as expanding along the second row, but with the signs switched. In general the "checkerboard" patterns below tells us how the signs go.

$$n \text{ odd: } \begin{vmatrix} + & - & + & \cdots & - & + \\ - & + & - & \cdots & + & - \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ + & - & + & \cdots & - & + \end{vmatrix}, \quad n \text{ even: } \begin{vmatrix} + & - & + & \cdots & + & - \\ - & + & - & \cdots & - & + \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ - & + & - & \cdots & + & - \end{vmatrix}.$$

Problem 1. Repeat the determinant calculations for the matrices A and B given in Problem 1 by expanding along different rows and columns.

9 Permutations Definition [Optional]

A **permutation** is a function that rearranges the elements of a list of numbers. Thus, $p : (1, 2, 3, 4, 5) \to (3, 2, 1, 5, 4)$ and $r : (1, 2, 3, 4, 5) \to (1, 2, 3, 5, 4)$

are permutations. In fact here p and r are just one-to-one functions from $\{1, 2, 3, 4, 5\}$ onto itself. Thus we may write $p(3) = 1$ or $r(5) = 4$. Permutations of the same list can be composed: $p \circ r : (1, 2, 3, 4, 5) \rightarrow (3, 2, 1, 4, 5)$.

The set of all possible permutations of the list $(1, 2, 3, \ldots, n)$ will be called $\mathcal{P}_n$.

Problem 1. List the elements of P_n for n equal to 2, 3, and 4.

Problem 2. Prove that $\mathcal{P}_n$ has $n!$ members. Hint: use induction.

The simplest permutation is one that just switches two entries in a list. Every permutation can be broken down into a sequence of switches. For example

$$(1, 2, 3, 4, 5) \rightarrow (3, 2, 1, 4, 5) \rightarrow (3, 2, 1, 5, 4)$$

and

$$(1, 2, 3, 4, 5) \rightarrow (2, 1, 3, 4, 5) \rightarrow (2, 3, 1, 4, 5) \rightarrow (2, 3, 1, 5, 4) \rightarrow (3, 2, 1, 5, 4)$$

are two ways to break p down into switches.

Problem 3. Consider $q : (1, 2, 3, 4, 5, 6) \longrightarrow (3, 4, 2, 1, 5, 6)$. Break q down into switches several different ways.

Problem 4. Prove that for any given permutation the number of switches used to create it is either always even or always odd. Hint, use induction. (*This result can be found in most introductory texts on group theory or abstract algebra.*)

The **parity** of a permutation is defined to be 0 if it decomposes into an even number of switches and 1 if it decomposes into an odd number of switches. The parity function is denoted by σ. Thus, $\sigma(p) = 0$ and $\sigma(q) = 1$.

We will be applying permutations to entries of matrices. For example if $A = [a_{ij}]^{6\times 6}$, then $a_{q(2)4} = a_{44}$ and $a_{q(1)q(6)} = a_{36}$.

We are now ready to present the alternative definition of determinants.

Definition 9.1. The **permutation determinant** of an $n \times n$ matrix is given by

$$\text{p-det}(A) = \sum_{p \in \mathcal{P}_n} (-1)^{\sigma(p)} a_{1p(1)} a_{2p(2)} a_{3p(3)} \cdots a_{np(n)}.$$

Check that for 3×3 matrices this becomes

$$\text{p-det}(A) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

Compare this to $\det(A)$; you should see that they are equal.

Theorem 9.2. *For any $n \times n$ matrix A we have $\det(A) = \text{p-det}(A)$.*

In other words the two definitions are equivalent and we shall use the term "permutation determinant" or the notation "p-det" after this is proved.

Outline of proof. Let $A = [a_{ij}]$ be an $n \times n$ matrix. We can use the uniqueness part of the proof of Theorem 5.1 to show

$$\det A = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_n=1}^{n} a_{1j_1} a_{2j_2} \cdots a_{nj_n} \det \begin{bmatrix} \mathbf{e}_{j_1} \\ \mathbf{e}_{j_2} \\ \vdots \\ \mathbf{e}_{j_n} \end{bmatrix}.$$

We know $\det \begin{bmatrix} \mathbf{e}_{j_1} \\ \vdots \\ \mathbf{e}_{j_n} \end{bmatrix} = 0$ if there are any repeated indices. You can check that if there are no repeated indices then $\det \begin{bmatrix} \mathbf{e}_{j_1} \\ \vdots \\ \mathbf{e}_{j_n} \end{bmatrix} = (-1)^{\sigma(j_1, j_2, \ldots, j_n)}$. But this proves the theorem! ☐

Problem 5. Find the determinants of A and B give in Problem 1 in Section 2 using this alternative definition of a determinant.

10 Cramer's Rule [Optional]

Cramer's Rule is a method for solving an $n \times n$ system of linear equations. It is based on another way of finding the inverse of a matrix. For larger systems it is an inefficient method. Row reduction to an upper triangular matrix is best.

However, if the matrix entries are variables or functions row reduction by hand may be messy. Cramer's Rule is often used in engineering and physics courses but is rarely used in math or computer science courses. We shall skip the proofs.

Definition 10.1. Given an $n \times n$ matrix A the ij **cofactor** is given by

$$c_{ij} = (-1)^{i+j} \det(A_{ij}),$$

where we recall that A_{ij} was obtained from A by deleting row i and column j.

The **adjoint matrix** of A is the matrix formed from its cofactors.

$$\operatorname{adj}(A) = [c_{ij}].$$

Theorem 10.2. *If* $\det(A) \neq 0$ *then*

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)}.$$

Problem 1. Use Theorem 10.2 to find the inverses of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} x & 2x & 3 \\ x^2 & 3 & 6x \\ 2-x & x & 0 \end{bmatrix}$$

Answers.

$$A^{-1} = \begin{bmatrix} \dfrac{d}{a\,d - b\,c} & \dfrac{-b}{a\,d - b\,c} \\ \dfrac{-c}{a\,d - b\,c} & \dfrac{a}{a\,d - b\,c} \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} \dfrac{2x^2}{D} & \dfrac{-x}{D} & \dfrac{-4x^2-3}{D} \\ \dfrac{2x^2-4x(x-2)}{D} & \dfrac{2-x}{D} & \dfrac{x^2}{D} \\ \dfrac{x^3+3x-6}{-3D} & \dfrac{3x^2-4x}{3D} & \dfrac{2x^3-3x}{3D} \end{bmatrix}$$

where $D = 5x^3 - 8x^2 - 3x + 6$. □

Theorem 10.3 (Cramer's Rule)**.** *Let* A *be an* $n \times n$ *nonsingular matrix and let* $\mathbf{b}$ *be an* $n \times 1$ *column vector. Define* $A(i, \mathbf{b})$ *to be the matrix obtained by replacing the* i*-th column of* A *with* $\mathbf{b}$*. Suppose* $\mathbf{v}$ *is the solution to* $A\mathbf{v} = \mathbf{b}$*. We can compute the* i*-th entry of* $\mathbf{v}$ *by*

$$v_i = \frac{\det A(i, \mathbf{b})}{\det A}.$$

Problem 2. Use Cramer's Rule to solve each system below.

(a) $$\begin{bmatrix}1 & 2 & 1\\ 2 & 2 & 1\\ 1 & 2 & 3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}5\\ 6\\ 9\end{bmatrix}$$

(b) $$\begin{bmatrix}x & 2x & 3\\ x^2 & 3 & 6x\\ 2-x & x & 0\end{bmatrix}\begin{bmatrix}f(x)\\ g(x)\\ h(x)\end{bmatrix}=\begin{bmatrix}1\\ 2\\ 0\end{bmatrix}$$

Answers. (a) $x = 1$, $y = 1$, and $z = 2$.
(b) Let $d(x) = 5x^2 - 3x - 6$. Then $f(x) = 2x/d(x)$, $g(x) = (2x - 4)/d(x)$ and $h(x) = -\frac{1}{3}(x^2 - 5x + 6)/d(x)$. $\square$

11 References

- Len Evans. *A Brief Course in Linear Algebra*, lecture notes used at Northwestern University.
- Charles W. Curtis. *Linear Algebra: An Introductory Approach*, 3^{rd} Edition, Allyn and Bacon, Inc. 1974.