Math 452
Homework Set 1: Solutions

This homework set is meant to be a review. All answers are to be written out in complete sentences with correct grammar and punctuation. It is due on the first day of class.

(1) Prove that $\sqrt{3}$ is not a rational number.

Suppose there are integers m and n such that $(\frac{m}{n})^2 = 3$. By the Prime Factorization Theorem, we may assume m and n have no common factors. In particular, we may assume they are not both divisible by 3.

Now $(\frac{m}{n})^2 = 3$ implies $m^2 = 3n^2$. Thus m^2 is divisible by 3. If m was not divisible by 3, then by the Prime Factorization Theorem, m^2 would not be divisible by 3. Thus we now know that m is divisible by 3. Hence we can write $m = 3k$ for some integer k.

Now $m^2 = 3n^2$ implies $9k^2 = 3n^2$ or $n^2 = 3k^2$. Thus n^2, and hence n, is divisible by 3.

But this contradicts that m and n can be selected so as not to have any common factors. Either, the Prime Factorization Theorem is wrong, or it is impossible for any m and n to have $(\frac{m}{n})^2 = 3$.

If you have not seen the proof of the Prime Factorization Theorem, look it up.

(2) Give a formal ϵ-δ style proof that $\lim_{x \to 2} 3x + 5 = 11$.

Let $\varepsilon > 0$ be given. Set $\delta = \frac{\varepsilon}{3}$.

Suppose x is such that $|x-2| < \frac{\varepsilon}{3}$.

It follows that $|3x-6| < \varepsilon$. Rewrite this using that $-6 = 5-11$, to get

$$|3x+5-11| < \varepsilon.$$

Therefore, by definition $\lim_{x \to 2} 3x+5 = 11$.

(3) Give a formal ϵ-δ style proof that $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

Suppose $\lim_{x \to 0} \sin\left(\frac{1}{x}\right) = L$. First, assume $L \geq 0$.

Let $\varepsilon = \frac{1}{2}$. Let $\delta > 0$. $\exists\, n \in \mathbb{N}$ s.t.

$$0 < \frac{1}{\frac{3\pi}{2} + 2\pi n} < \delta.$$

Why? Here is a proof: $\exists\, n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < \delta$. Then

$$0 < \frac{1}{\frac{3\pi}{2} + 2\pi n} < \frac{1}{2\pi n} < \frac{1}{n} < \delta.$$

Next let $x_n = \frac{1}{\frac{3\pi}{2} + 2\pi n}$ for every $n \in \mathbb{N}$. Notice that $\forall\, n \in \mathbb{N}$ $\sin\left(\frac{1}{x_n}\right) = -1$. Thus $\forall\, \delta > 0$, $\exists\, n \in \mathbb{N}$ s.t.

$$\left| \sin\left(\tfrac{1}{x_n}\right) - L \right| > \tfrac{1}{2} = \varepsilon,$$

even though $|x_n - 0| < \delta$. Thus it cannot be that $L \geq 0$.

Assum $L < 0$. Let $x_n = \frac{1}{\frac{\pi}{2} + 2\pi n}$ and ~~repeat~~ repeat the argument above. Again $\left|\sin\left(\frac{1}{x_n}\right) - L\right| = |1 - L| > \frac{1}{2}$ since $L < 0$.

Thus $L \not< 0$ and $L \neq 0$ and $L \not> 0$. This contradicts the Trichotomy Property of the real numbers.

(See pg 15 in Pugh's textbook.)

(4) Define $f(x)$ to be $x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and 0 for $x = 0$. Is f differentiable at $x = 0$? Prove your claim.

By definition, $f'(x) = \lim\limits_{h \to 0} \dfrac{(x+h)^2 \sin\left(\frac{1}{x+h}\right) - x^2 \sin\frac{1}{x}}{h}$.

Let $x = 0$. We have $f'(0) = \lim\limits_{h \to 0} \dfrac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = \lim\limits_{h \to 0} h \sin\frac{1}{h}$.

We can apply the Squeeze Theorem, see ~~Stewar~~ Stewart's Essential Calculus, 2nd edition, pg 41. Since $-1 \le \sin\frac{1}{h} \le 1$ for positive h we get $-h \le h\sin\frac{1}{h} \le h$. Since $\lim\limits_{h \to 0} h = 0$ and $\lim\limits_{h \to 0} -h = 0$, we get $\lim\limits_{h \to 0^+} h \sin\frac{1}{h} = 0$. A similar argument works for $h < 0$. Thus $f'(0)$ exists and equals 0.

Now, here is something to think about. For $x \neq 0$,

$f'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$. But $\lim\limits_{x \to 0} 2x \sin\frac{1}{x} - \cos\frac{1}{x}$ does not exist! Check this. We conclude that $f'(x)$ exists, but is not continuous at $x=0$.

(5) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Prove that if f has a local maximum or minimum at c then $f'(c) = 0$.

This result is called Fermat's Theorem.

See Stewart's Essential Calculus, 2nd edition, pages 205-206.

(6) Use the Intermediate Value Theorem, the Mean Value Theorem, and other facts, to prove that the function

$$f(x) = 4x^5 + x^3 + 2x + 1$$

has one and only one real zero.

Observe that $f(0)=1$ and $f(-1)=-6$. Since polynomials are continuous we can apply the I.V.T. to show that $f(c)=0$ for at least one value $c \in (-1,0)$.

Suppose there is another value $a \in \mathbb{R}, a \neq c$, where $f(a)=0$. Since f is a polynomial it is differentable, so we can apply the MVT on $[c,a]$ if $a>c$ or $[a,c]$ if $a<c$.

Suppose $a>c$, the other case being similar.

$f'(x) = 20x^4 + 3x^2 + 2$. For all values of x x^4 and $x^2 \geq 0$. Thus $f'(x) \geq 2$ $\forall x \in \mathbb{R}$.

In particular $f'(x)$ is never zero. But the MVT says that $\exists\, b \in (c,a)$ s.t.

$$f'(b) = \frac{f(a)-f(c)}{a-c} = \frac{0}{a-c} = 0.$$

This contradiction shows that $a \not> c$. The case $a<c$ is similar. Thus c is the only value for which f is zero.

(7) Prove that the sum of the first n positive odd integers is equal to n^2.

For $n=1$, we have that $1 = 1^2$. Suppose for a fixed value k we knew that $\sum_{i=1}^{k} (2i-1) = k^2$.

Consider the sum $\sum_{i=1}^{k+1} (2i-1)$. We compute

$$\sum_{i=1}^{k+1} (2i-1) = \sum_{i=1}^{k} (2i-1) + 2(k+1) - 1 = k^2 + 2k + 1 = (k+1)^2.$$

Thus, by the Principle of Mathematical Induction

$$\sum_{i=1}^{n} (2i-1) = n^2$$

for all positive integers n.

(8) Evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} (x_i)^2 \,\Delta x$, where $x_i = 1 + i/n$ and $\Delta x = 1/n$.

Do this directly without using integration.

Substiting $x_i = 1+\frac{i}{n}$ and $\Delta x = \frac{1}{n}$ we get the sum $\sum_{i=1}^{n}(1+\frac{i}{n})^2\frac{1}{n}$.

Now,

$$S_n = \sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^2\frac{1}{n} = \frac{1}{n}\left(\sum_{i=1}^{n} 1 + \frac{2i}{n} + \frac{i^2}{n^2}\right) = \frac{1}{n}\left(\sum_{i=1}^{n} 1 + \frac{2}{n}\sum_{i=1}^{n} i + \frac{1}{n^2}\sum_{i=1}^{n} i^2\right).$$

Using standard summation formulas, see Stewart, Appendix B, we have

$$S_n = \frac{1}{n}\left(n + \frac{2}{n}\left(\frac{n(n+1)}{2}\right) + \frac{1}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right) =$$

$$1 + \frac{n+1}{n} + \frac{(n+1)(2n+1)}{6n^2} = 1+1+\frac{1}{n}+\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2} = \frac{7}{3}+\frac{3}{2n}+\frac{1}{6n^2}.$$

Now we see that $\lim_{n\to\infty} S_n = \frac{7}{3}$.

Notice $\lim_{n\to\infty} S_n = \int_1^2 x^2\,dx = \frac{x^3}{3}\Big|_1^2 = \frac{8}{3}-\frac{1}{3} = \frac{7}{3}$.

(9) Express $\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^3+x_i\sin(x_i))\Delta x$, where $x_i = i\pi/n$ and $\Delta x = \pi/n$, as a definite integral, then evaluate it.

The Riemann sum $\sum_{i=1}^{n} f(x_i)\Delta x$, $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$, corresponds, in the limit as $n\to\infty$, to the integral $\int_a^b f(x)\,dx$.

We have $x_i = i\Delta x$, so $a=0$. Then $\Delta x = \frac{b}{n} = \frac{\pi}{n}$. Hence $b=\pi$.

Thus, the limit given is equal to $\int_0^\pi x^3 + x\sin x\,dx$.

Now $\int_0^\pi x^3 + x\sin x\,dx = \int_0^\pi x^3\,dx + \int_0^\pi x\sin x\,dx$.

By the Fundamental Theorem of Calculus $\int_0^\pi x^3\,dx = \frac{x^4}{4}\Big|_0^\pi = \frac{\pi^4}{4}$.

The Integration by Parts Formula says $\int u\,dv = uv - \int v\,du$.

Let $u=x$ and $dv = \sin x\,dx$. Then $du = dx$ and $v = -\cos x$.

Thus,

$$\int_0^\pi x\sin x\,dx = -x\cos x\Big|_0^\pi + \int_0^\pi \cos x\,dx = \pi - 0 + \sin x\Big|_0^\pi = \pi.$$

We conclude that

$$\int_0^\pi x^3 + x\sin x\,dx = \frac{\pi^4}{4} + \pi.$$

(10) Prove that $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ when $|r| < 1$.

First we shall show that $\sum_{n=0}^{p} ar^n = \frac{a(1-r^{p+1})}{1-r}$.

Let $S_p = \sum_{n=0}^{p} ar^n = a + ar + ar^2 + ar^3 + \cdots + ar^p$.

Then $rS_p = ar + ar^2 + ar^3 + ar^4 + \cdots ar^{p+1}$.

Observe that $S_p - rS_p = a - ar^{p+1} = a(1-r^{p+1})$.

Solving for S_p gives

$$S_p = \frac{a(1-r^{p+1})}{1-r}.$$

Now we compute the limit.

$$\lim_{p \to \infty} S_p = \lim_{p \to \infty} \frac{a(1-r^{p+1})}{1-r} = \frac{a}{1-r}$$

Since $\lim_{p \to \infty} r^{p+1} = 0$ for $|r| < 1$.

(11) Find the intervals of convergence for the two series below.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{5^n n^5}$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt[4]{n}}$

a. We will use the Ratio Test. Let $a_n = \frac{x^n}{5^n n^5}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{5^{n+1}(n+1)^5} \cdot \frac{5^n n^5}{x^n}\right| = \frac{|x|}{5} \cdot \frac{n^5}{(n+1)^5}.$$

Now, $\displaystyle\lim_{n\to\infty} \frac{n^5}{n+1^5} = \left(\lim_{n\to\infty} \frac{n}{n+1}\right)^5 = 1^5 = 1$. ~~Thus,~~ Thus,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{5}.$$

By the Ratio Test we have convergence on $(-5,5)$ and divergence on $(-\infty,-5)\cup(5,\infty)$. We just need to check the end points.

Let $x=-5$. Then $a_n = \frac{(-1)^n}{n^5}$. By the Alternating Series Test $\sum_{n=1}^{\infty} a_n$ converges.

Let $x=5$. Then $a_n = \frac{1}{n^5}$. By the p-series Test $\sum_{n=1}^{\infty} a_n$ converges.

Thus the interval of convergence is $[-5,5]$.

b. Again we use the Ratio Test. Let $a_n = \frac{(-2)^n x^n}{n^{\frac{1}{4}}}$. Thus

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-2)^{n+1} x^{n+1}}{(n+1)^{1/4}} \cdot \frac{n^{\frac{1}{4}}}{(-2)^n x^n}\right| = \left|-2x \cdot \frac{n^{\frac{1}{4}}}{(n+1)^{\frac{1}{4}}}\right| = 2|x|\left(\frac{n}{n+1}\right)^{\frac{1}{4}}.$$

$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{\frac{1}{4}} = \left(\lim_{n\to\infty} \frac{n}{n+1}\right)^{1/4} = 1^{1/4} = 1$. Thus, by the Ratio Test the series converges on $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and diverges on $\left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$. We check the end points.

Let $x = -\frac{1}{2}$. Then $a_n = \frac{1}{n^{1/4}}$. By the p-series Test $\sum a_n$ diverges.

Let $x = \frac{1}{2}$. Now $a_n = \frac{(-1)^n}{n^{1/4}}$. By the Alternating Series Test $\sum a_n$ converges.

Hence, the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right]$.

(12) Evaluate $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$. Hint: Taylor series.

Recall that the Taylor series for arctan(x), centered at $x=0$, is

$$S(x) = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}(-1)^{n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad (*)$$

If we evaluate at $x=1$ we get $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

By the Alternating Series Test, this series converges.

By Taylor's Theorem we know that on $x \in (-1,1)$ the power series $(*)$ converges to $\arctan(x)$.

It is tempting to write

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots = \arctan(1) = \frac{\pi}{4}.$$

The problem is we don't know that $S(x)$ is continuous at $x=1$.

Recall that $\lim_{n\to\infty} x^n$ on $[0,1]$ is the discontinuous function $f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x=1 \end{cases}$.

However, Abel Theorem (see Abbott's "Understanding Analysis," pg 172) tells us that the convergence in our case is uniform on $[0,1]$ and hence does give a continuous limit. Thus we can now conclude that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \arctan(1) = \frac{\pi}{4}.$$