Properties of the Real Number System

- Algebraic Properties.

  $\mathbb{R}$ is an ordered field, with $\mathbb{Q}$ an ordered subfield, and $\mathbb{Z}$ an ordered communative subring with a unit.
- Cardnality.

  $\mathbb{R}$ is uncountably infinite, while $\mathbb{Q}$ and $\mathbb{Z}$ are countably infinite.
- Dense Subsets.

  Between any pair of distinct real numbers there is a rational number and an irrational number.
- Archimedean Properties.

  For every $x \in \mathbb{R}$ there exists an $n \in \mathbb{N}$ such that $n > x$.
  For every $x \in (0, \infty)$ there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.
- ϵ-Principle.

  If $\forall \epsilon > 0$ we have $a \leq b + \epsilon$, then $a \leq b$.
  If $\forall \epsilon > 0$ we have $|x - y| \leq \epsilon$, then $x = y$.
- Completeness Properties.

  If $S \subset \mathbb{R}$ has an upper bound, then S has a least upper bound; it is called the *supremum* of S and is denoted $\sup S$. If $S \neq \emptyset$ and has no upper bound then we define $\sup S = \infty$. We define $\sup \emptyset = -\infty$.

  If $S \subset \mathbb{R}$ has a lower bound, then S has a greatest lower bound; it is called the *infimum* of S and is denoted $\inf S$. If $S \neq \emptyset$ and has no lower bound then we define $\inf S = -\infty$. We define $\inf \emptyset = \infty$.
- Subsets.

  If $\emptyset \neq S \subset T$, then
  $$-\infty \leq \inf T \leq \inf S \leq \sup S \leq \sup T \leq \infty.$$
- Cauchy Completeness.

  An infinite sequence of real numbers (a_n) converges iff
  $$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \, \text{such that} \, n, m \geq N \implies |a_n - a_m| < \epsilon.$$
- Existance of Roots.

  For every $x \in [0, \infty)$ and $n \in \mathbb{N}$, there exists a unique $y \in [0, \infty)$ such that $y^n = x$.
- Triangle Inequality.

  In $\mathbb{R}^n$ we have $|x - z| \leq |x - y| + |y - z|$.