Def A <u>metric space</u> is a set M and a function

$$d: M \times M \to [0,\infty) \quad \text{s.t.}$$

1. $d(x,y) = 0$ iff $x = y$. pos. definite.
2. $d(x,y) = d(y,x)$ symmetric
3. $d(x,z) \leq d(x,y) + d(y,z)$ triangle ineq.

Ex $\mathbb{R}$ with $d(x,y) = |x-y|$.
$\mathbb{Q}$ with $d(x,y) = |x-y|$.
$\mathbb{Z}$ with $d(x,y) = |x-y|$
$\mathbb{Z}$ with $d'(x,y) = 1$ $x \neq y$ 0 $x = y$.

On $\mathbb{R}^n$, the usual metric is $\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$.

Ex On $\mathbb{R}^2$ graph the "unit circle" for the metrics

$$d_n((x_1,y_1),(x_2,y_2)) = \sqrt[n]{|x_1 - x_2|^n + |y_1 - y_2|^n}$$

and $d_\infty = \sup(|x_1 - x_2|, |y_1 - y_2|)$

Unit circles in R^2 uisng different metrics

```
> with(plots):

> ball1:=implicitplot(
  abs(x)+abs(y)=1,x=-1..1,y=-1..1,thickness=2,color=plum,numpoints=1000):

> ball2:=implicitplot( x^2+y^2=1,x=-1..1,y=-1..1,thickness=2,color=red):

> ball3:=implicitplot( abs(x^3)+abs(y^3)=1,x=-1..1,y=-1..1,thickness=2,color=green):

> ball4:=implicitplot( x^4+y^4=1,x=-1..1,y=-1..1,thickness=2,color=blue):

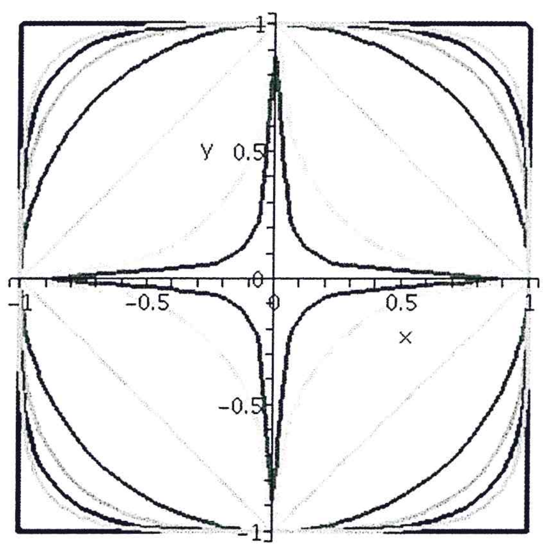
> ball5:=implicitplot( abs(x^5)+abs(y^5)=1,x=-1..1,y=-1..1,thickness=2,color=cyan):

> ballhalf:=implicitplot(
  sqrt(abs(x))+sqrt(abs(y))=1,x=-1..1,y=-1..1,thickness=2,color=pink,numpoints=1000):

> ballthird:=implicitplot(
  root(abs(x),3)+root(abs(y),3)=1,x=-1..1,y=-1..1,thickness=2,color=brown,numpoints=1000):

> ballinfinity:=implicitplot(
  max(abs(x),abs(y))=1,x=-1..1,y=-1..1,thickness=2,color=black,numpoints=10000):

> display(ballinfinity,ballthird,ballhalf,ball1,ball2,ball3,ball4,ball5);
```

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>
```

Def A sequence (a_n) converges to a if

$\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow d(a_n, a) < \varepsilon$.

Fact Limits, when they exist, are unique. That is if $a_n \to a$ and $a_n \to b$, then $a = b$.

Pf Let $\varepsilon > 0$
Let N be s.t. $n \geq N \Rightarrow d(a_n, a) < \frac{\varepsilon}{2}$

Let M be s.t. $m \geq M \Rightarrow d(a_m, b) < \varepsilon/2$.

~~Let~~ Then for all $n \geq \max(N, M)$

$$d(a,b) \leq d(a, a_n) + d(a_n, b) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\forall \varepsilon > 0$, $d(a,b) < \varepsilon$. By the ε-principle (see pg 21) $d(a,b) = 0$ and so $a = b$. ■

Thm Every subseq of a convergent seq converges to the same limit.

Pf: See textbook, pg 54.

Continuity

Def Let (M, d) and (N, d') be metric spaces. Let $f: M \to N$ be a function. f is cont. at $x \in M$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $d(x,y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon$.

f is cont. on M if it is cont. at each $x \in M$.

Thm $f: M \to N$ is cont. on M iff $a_n \to a \Rightarrow f(a_n) \to f(a)$.

Pf Let f be cont at a, and $a_n \to a$. We will show $f(a_n) \to f(a)$. Let $\varepsilon > 0$. $\exists \delta > 0$ s.t. $d(a,y) < \delta \Rightarrow d'(f(a), f(y)) < \varepsilon$. $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow d(a, a_n) < \delta$. Thus $d'(f(a), f(a_n)) < \varepsilon$ for $n \geq N$. Hence $f(a_n) \to f(a)$.

Let $a \in M$. Assume $a_n \to a \Rightarrow f(a_n) \to f(a)$. We will suppose f is not cont. at a, and derive a contradiction. $\exists\ \varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists\ x \in M$ with $d(a,x) < \delta$ but $d'(f(a), f(a_n)) \geq \varepsilon_0$.

Thus for each $n \in \mathbb{N}$, $\exists\ a_n \in B(a, \frac{1}{n})$ s.t. $d(f(a), f(a_n)) \geq \varepsilon_0$. Thus $a_n \to a$, but $f(a_n) \not\to f(a)$. ▨

Corollary If $f: M \to N$ and $g: N \to P$ are cont. so is $g \circ f: M \to P$.

Pf: See textbook.

Def Let $f: M \to N$ be cont., one-to-one, onto, with $f^{-1}: N \to M$ cont. Then f is a <u>homeomorphism</u> and M and N are <u>homeomorphic</u>. This is an eq. rel. on the class of metric spaces.

Ex $(0,1) \approx (0,2)$. $(0,1) \approx \mathbb{R}$. $[0,1] \approx [0,2]$, $[0,1)$, $(0,3]$. But $[0,1)$, $(0,1)$, $[0,1]$ are not as we will see later.

Open and Closed Sets

Def Let $U \subset M$. If $\forall p \in U$, $\exists\, \varepsilon > 0$ s.t. $B(p, \varepsilon) \subset U$, then U is open. Equivalently, U is open iff it is a union of open balls.

Def Let S be a subset of a metric sp M. Let $x \in M$. If $\exists\, (s_n)$ in S s.t. $s_n \to x$, we say x is a limit (or limit pt) of S. Let $\lim S$ = all limit pts of S.

Def If $\lim C \subset C$, then C is closed. Equivalently, C is closed iff $M - C$ is open.

Def Let $x \in M$. A <u>nbhd</u> of x is an open set containing x. Sometimes I'll say "open nbhd" but this is redundant. A <u>closed nbhd</u> of x, is a closed set C that contains an open nbhd of x.

Fact Let $f: M \to N$. f is cont. at x iff $\forall$ nbhd of y (N_y) $\exists$ a nbhd of x (N_x) s.t. $f(N_x) \subset N_y$.

Def The collection of all open subset of M is called the <u>topology</u> of M.

Thm Let $\mathcal{T}$ be the top. of M. Then...

(a) Every union of open sets is an open set
(b) The intersection of a finite collection of open sets is open.
(c) M and ϕ are open.

Pf We did (c). see textbook. It is easy.

Corollary Dual statements are true for closed sets. Every intersection of closed sets is closed and finite unions of closed sets are closed.

Examples $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ is not open

$\bigcup_{n=2}^{\infty} \left[\frac{1}{n}, 1\right] = (0, 1]$ is not closed.

Open Subsets of $\mathbb{R}$ [1]

Definition. $(-\infty, a)$, (a, ∞), $(-\infty, \infty)$, (a, b) are the *open intervals* of $\mathbb{R}$. (Note that these are the *connected* open subsets of $\mathbb{R}$.)

Theorem. Every open subset U of $\mathbb{R}$ can be uniquely expressed as a countable union of disjoint open intervals. The end points of the intervals do not belong to U.

Proof. Let $U \subset \mathbb{R}$ be open. For each $x \in U$ we will find the "maximal" open interval I_x s.t. $x \in I_x \subset U$. Here "maximal" means that for any open interval J s.t $x \in J \subset U$, we have $J \subset I_x$.

Let $x \in U$. Define $I_x = (a_x, b_x)$, where $a_x = \inf \{a \in \mathbb{R} \,|\, (a, x) \subset U\}$, and $b_x = \sup \{b \in \mathbb{R} \,|\, (x, b) \subset U\}$. Either could be infinite. They are well defined since $\exists$ an open interval I s.t $x \in I \subset U$, because U is open.

Clearly, $x \in I_x \subset U$. Suppose $J = (p, q)$ is s.t. $x \in J \subset U$. Then $(p, x) \subset U$ so $p \geq a_x$. Likewise $q \leq b_x$. Thus, $J \subset I_x$ and so I_x is indeed maximal.

Suppose $a_x \in U$. (In particular we are supposing a_x is finite.) Then $\exists\ \epsilon > 0$ s.t. $(a_x - \epsilon, a_x + \epsilon) \subset U$. The $(a_x - \epsilon, b_x)$ is larger than I_x contradicting maximality. Thus, $a_x \notin U$ and likewise $b_x \notin U$.

Claim. For $x, y \in U$, the intervals I_x and I_y are either disjoint or identical. **Proof.** If $I_x \cap I_y \neq \emptyset$, then $I_x \cup I_y$ is an open interval. Since $x \in I_x \cup I_y \subset U$, we have, by maximality, $I_x \cup I_y \subset I_x$. Likewise, $I_x \cup I_y \subset I_y$. Thus, by elementary set theory, $I_x = I_y$. $\Box$

We now have that U is a disjoint union of maximal open intervals. Call this collection $\mathcal{I}$. How do we know there are at most countably many distinct members of $\mathcal{I}$? Remember that the rational numbers are a countable *dense* subset of $\mathbb{R}$.

For each distinct I_x choose a rational point in I_x. Because these intervals are disjoint this determines a one-to-one map from $\mathcal{I}$ into $\mathbb{Q}$. Hence $\mathcal{I}$ is finite or countable. (See Chapter 1, Section 4 on cardinality.)

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[1] ©2014, Michael C. Sullivan, may be used for noncommercial educational purposes. Based on the textbook *Real Mathematical Analysis* by Charles Pugh.

Now for uniqueness. Suppose $\mathcal{J}$ is a collection of disjoint open intervals whose union is U. Let $J = (a, b) \in \mathcal{J}$ and $x \in J$. We know there is an $I_x \in \mathcal{I}$. Clearly $I_x \cap J \neq \emptyset$. Since I_x is maximal, $J \subset I_x$. We claim that $J = I_x$. Suppose not. Either $a_x < a$ or $b < b_x$. We assume the latter as both cases are similar. It follows that b is finite. Now $b \notin J$, but $b \in I_x \subset U$. Thus, $\exists\ J' \in \mathcal{J}$ s.t. $b \in J'$. Now, $\exists \epsilon > 0$ s.t. $(b - \epsilon, b + \epsilon) \subset J'$. But then $\exists 0 < \delta < \epsilon$ s.t. $b - \delta$ is in J and J'; hence they are not disjoint. Thus, $J = I_x$.

Now we have $\mathcal{J} \subset \mathcal{I}$. If $\mathcal{J} \neq \mathcal{I}$ then $\mathcal{J}$ cannot have union all of U.

$\square$

Remark. This result does not hold in all metric spaces. Indeed to make sense of it we would need a concept analogous to the open intervals. Even in $\mathbb{R}^2$ it is easy to draw open sets that are not the disjoint union of open balls. But, part of this result can be generalized to $\mathbb{R}^n$.

Theorem. Let $\mathcal{U}$ be a collection of disjoint open subsets of $\mathbb{R}^n$. Then $\mathcal{U}$ is at most countable.

Outline of Proof. Let $U \in \mathcal{U}$. Let $x \in U$. Let B be an open ball s.t. $x \in B \subset U$. Show that $\exists\ y \in B$ with rational coordinates. This determines a one-to-one map from $\mathcal{U}$ into $\mathbb{Q}^n$, a countable set. $\square$

Definition. Let M be a metric space. If M contains a countable dense subset, we say M is a *separable space* and the Theorem above holds. See pages 128-129 of the textbook. Later we will define what it means for a subset of a metric space to be *connected*. Then a further generalization is possible.

<u>Other Characterizations of Continuity.</u>

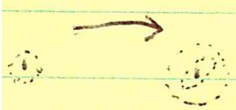
<u>Thm (10)</u> Let $f: M \to N$. The following are equivalent.

1. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$.
2. f^{-1} takes closed sets to closed sets.
3. f^{-1} takes open sets to open sets.

<u>Pf</u> ① $\Rightarrow$ ②. Assume ①. Let $C \subset N$ be closed. Let $D = f^{-1}(C) = \{x \in M \mid f(x) \in C\}$. To show D is closed consider a seq (p_n), $p_n \in D$, with $p_n \to p \in M$. We will show $p \in D$.

We know $f(p_n) \to f(p)$. Since all $f(p_n) \in C$ and C is closed, we know $f(p) \in C$. Hence $p \in D$. Thus D is closed.

② $\Rightarrow$ ③. Assume ②. Let $U \subset N$ be open. $f^{-1}(U)$ is open is equivalent to $(f^{-1}(U))^c$ is closed. But



$$(f^{-1}(U))^c = f^{-1}(U^c)$$, which is closed by assumption.

Draw Picture.

③ $\Rightarrow$ ①. Assume ③. Let $\varepsilon > 0$. Let $x \in M$. Let B = open ball $B(f(x), \varepsilon)$. Let $U = f^{-1}(B)$. It is open. ~~Let~~ Let $\delta > 0$ be s.t. open $B(x, \delta) \subset U$. Then $d(x,y) < \delta \Rightarrow y \in B(x,\delta) \Rightarrow f(y) \in B(f(x), \varepsilon) \Leftrightarrow d(f(x), f(y)) < \varepsilon$. ▨

This motivates the following idea.

Def Let X be a set and let $\mathcal{T}$ be a collection of subsets of X. Suppose $\mathcal{T}$ has the following properties.

(a) The union of any subcollection of sets in $\mathcal{T}$ is also in $\mathcal{T}$.

(b) The intersection of any finite subcollection of sets in $\mathcal{T}$ is also in $\mathcal{T}$.

(c) $X, \phi \in \mathcal{T}$.

Then $(X, \mathcal{T})$ is called a topological space and ~~the $\mathcal{T}$ is~~ the set in $\mathcal{T}$ are called the open sets.

It is now possible to define convergence and continuity without reference to a metric.

Back the metric spaces.

Closure, Interior, and Boundary.

Let S be a subset of a metric space M,

$cl(S) = \bar{S} = \bigcap K$, where $S \subset K$, and K is closed.

Fact: $\bar{S} = \lim S$. See textbook for proof.

$int(S) = S^{\circ} = \bigcup U$ where $U \subset S$, U is open.

Frontier S = $Fr(S) = \bar{S} \cap \overline{S^c}$. Also called boundary and denote ∂S, but this can conflict with manifold terminology.

$Fr(\mathbb{Q}) = \mathbb{R}$.

Subspaces: Let (M, d) be a metric space. Let $N \subset M$. d gives a metric on N and so (N, d) is a metric space and is called a subspace of M.

Suppose $U \subset N \subset M$, is open in N. Is it open in M? Not necessarily.

Ex. $[0,1] \subset \mathbb{R}$ is a ~~subspa~~ subspace. Then $[0,1]$ ~~[illegible]~~ is open as a subset of $[0,1]$, but not of $\mathbb{R}$.

~~The~~ book gives ~~[illegible]~~ others examples.

There are conditions under which openness or closedness are "inherited".

<u>Facts</u>: If $N \subset M$ is closed, then $K \subset N$ closed $\Leftrightarrow$ $K \subset M$ closed.

If $N \subset M$ is open, then $U \subset N$ open $\Leftrightarrow$ $U \subset M$ open.

<u>Pf</u>: Let $N \subset M$ be closed. If $K \overset{\subset N}{\text{is}}$ closed in M, $K = K \cap N$ is closed in N. Suppose $K \subset N$ is closed. Then $\lim_N K = K$. If $\lim_M K$ has a pt not in N then $\lim N \neq N$. ~~since~~

Let $N \subset M$ be open. You can either do a direct proof and work with complements.

A more general result holds.

<u>Thm</u> Let $K \subset N \subset M$. $K \subset N$ is closed iff $\exists$ closed set $L \subset M$ s.t. $K = N \cap L$. Likewise for open sets.

<u>Pf</u> See pg 68-69.

Clustering and Condensing

Def Let M be a metric space, $S \subset M$, $p \in M$.

p is a <u>cluster pt</u> of S if $\forall \varepsilon > 0$ $B(p,\varepsilon) \cap S$ is infinite.

p is a <u>condensation pt</u> of S if $\forall \varepsilon > 0$, $B(p,\varepsilon) \cap S$ is uncountable.

Recall: p is a <u>limit pt</u> of S if $\forall \varepsilon > 0$, $B(p,r) \cap S \neq \phi$.
(see Lemma 8 by 62.)

Obviously, a cond. pt is a cluster pt. is a limit pt.

Def Let S' denote the set of cluster points of S.

Facts $\bar{S} = S \cup S' = \lim S$. S is closed iff $S' \subset S$.
If $S \subset \mathbb{R}$ is $\neq \phi$, then $\operatorname{lub} S$, $\operatorname{glb} S \in \bar{S}$.

Proofs are elementary. See textbook.

Product Metrics

Def Let $M = M_1 \times M_2$, where (M_1, d_1) and (M_2, d_2) are metric spaces. Then the Euclidean product metric for M is

$$d(p,q) = \sqrt{d_1(p_1,q_1)^2 + d_2(p_2,q_2)^2}$$

Two other useful metrics are

$$d_{max} = \max\{d_1(p_1,q_1),\ d_2(p_2,q_2)\}$$

$$d_{sum} = d_1(p_1,q_1) + d_2(p_2,q_2).$$

You should check these are metrics, see Exercise 83, pg 124.

Ex For $\mathbb{R}^2$ we drew the unit circle in each of these.

Rmk Generalizing to finite products is straight forward. Generalizing to infinite products is somewhat subtle.

See textbook for other useful facts, such as

$$d_{max} \le d \le d_{sum} \le 2d_{max}.$$

Arithmetic in $\mathbb{R}$

The standard opperations in $\mathbb{R}$ are cont. See text.

Completeness

Cauchy sequences are defined as before.
A metric space is <u>complete</u> if C-seq.'s converge.

<u>Fact</u> $\mathbb{R}^n$ is complete as are closed subspaces.

<u>Q:</u> Given a metric sp. that is not complete, can we "add some points" and complete it? Is this process unique?

Ex: $\mathbb{Q} \to \mathbb{R}$

$(0,1) \to [0,1]$

<u>Fact</u> If M and N are homeo. either both are complete or neither ~~is~~ complete.

Boundedness

<u>Def</u> A subset S of a metric space M is bdd if $\exists\, r>0$, $p \in M$ s.t. $S \subset B(p,r)$.

Bddness is not a topological property.

Compactness

There are several ways of defining compactness in a metric space that are equivalent. But in general top. sp's they are not eq. In $\mathbb{R}^n$ these are eq. to being closed and bounded.

Def $S \subset M$ is <u>sequentially compact</u> if every infinite seq has a convergent subseq. with limit in S.

Thm In a metric space every seq. comp. subset is closed and bounded.

pf Know this!

Let $S \subset M$ be seq. comp. Let p be a limit point of S. Let $(a_n) \subset S$ be $a_n \to p$. Let a_{n_k} be a convergent sub seq. with $a_{n_k} \to q \in A$. But then $p = q$ (see Thm 1 page 54). Thus $p \in A$ and A is closed by def. (pg 59).

Suppose S is not bounded. Then let $p \in M$ and $B_n = B(p, n)$. $\forall n \in \mathbb{N}$ $\exists$ $a_n \in B_n \setminus S$. But, you can show, (a_n) does not have a convergent subseq. ∎

Fact $[a,b] \subset \mathbb{R}$ seq compact. $\square$

Thm The Cartesian product of two seq compact sets is seq comp.

Pf Let $A \subset M$, $B \subset N$ where M, N are metric spaces, ~~Consider~~ and A, B are seq. comp. Consider $A \times B \subset M \times N$ using the E. prod. metric.

Let $s_n = (a_n, b_n) \subset A \times B$ be a sequence. Suppose $a_{n_k} \to a$ in A. The subseq b_{n_k} in B has a subseq $b_{n_{k_\ell}} \to b$ in B. Since $a_{n_{k_\ell}} \to a$

we have $s_{n_{k_\ell}} \to (a,b)$. Thus $A \times B$ is seq. comp.

Fact: Easy: This generalizes to products of ~~~~ n ~~sets~~ seq. comp sets by induction.

Hard: This generalizes to infinite products, with the "right" definitions. This result is called the Tychonoff Theorem. See Ch 5 of Topology by Munkres. We cover this in 530.

Note to self: countable products of seq. comp. sp's. are seq. comp. (Kelly, pg 238, D(a)). But the uncountable prod. of I is not (Willard, 17G(1)).

Bolzano-Weierstrass Thm

Any bdd seq in $\mathbb{R}^n$ has a convergent subseq.

Pf A box $[a_1,b_1]\times\cdots\times[a_n,b_n] \subset \mathbb{R}^n$ is seq. comp.
A bdd seq. is thus in some seq. comp. set and so has a conv. subseq.

Thm A closed subset of a seq. comp. ~~seq~~ set is seq. comp.

Heine-Borel Thm

Every closed bdd subset of $\mathbb{R}^m$ is seq. comp.

Pf: Obvious. Know these three.

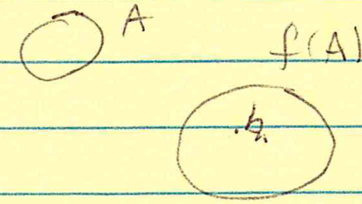
Thm The intersection of a nested seq of compact non-empty sets is compact & not empty.

Ex $\bigcap_{n=1}^{\infty} [n,\infty) = \phi$. $\bigcap_{n=2}^{\infty} (1-\frac{1}{n},1) = \phi$.

Continuity and Compactness

Thm (3.4) If $f: M \to N$ is cont. and $A \subset M$ is compact then $f(A)$ is compact in N.

Pf Let (b_n) be a seq. in $f(A)$.



Let $a_n \in f^{-1}(b_n)$.

$\exists$ conv. subseq (a_{n_k}) in A.
Let $a_{n_k} \to a$.

Then $b_{n_k} \to f(a)$.

$\blacksquare$

Fact If M & N are homeo, ~~are~~ either both are compact or neither are.

Note: Cont image of a closed set need not be closed.
$\arctan([1, \infty)) = [\frac{\pi}{4}, \frac{\pi}{2})$
Cont. image of a bdd set need not be bdd.

But the cont. image of a closed bdd set is closed and bdd.

You know that a cont. func. on ~~a~~ $[a,b] \to \mathbb{R}$ obtains its max/min (Extreme Value Thm). Let $f: M \to \mathbb{R}$. Let $K \subset M$ be compact. Then f obtains its max and min over K. See text.

You know (kind of) that a cont. func on $[a,b]$ is uniformly cont. This holds more generally: if $f: K \to \mathbb{R}$ is cont. & K is compact, then f is unif. cont.

Pf Suppose not. $\exists\, \varepsilon > 0$ s.t. $\forall\, \delta > 0$ $\exists\, p, q \in K$ with $d(p,q) < \delta$ but $|f(p) - f(q)| \geq \varepsilon$.

Let $\delta = \frac{1}{n}$, $n = 1, 2, 3 \ldots$ For each $n \in \mathbb{N}$ select p_n, q_n s.t. $d(p_n, q_n) < \delta = \frac{1}{n}$ but $|f(p_n) - f(q_n)| \geq \varepsilon$.

By compactness, $\exists$ subsequences $p_{n_k} \to p \in K$.

Since $d(p_n, q_n) \to 0$ $d(p_{n_k}, q_{n_k}) \to 0$ and

$q_{n_k} \to p$. (~~Suppose not~~ $d(p, q_{n_k}) \leq d(p, p_{n_k}) + d(p_{n_k}, q_{n_k})$
$\to 0 + 0 = 0$.)

By cont. $f(p_{n_k}) \to f(p)$ and $f(q_{n_k}) \to f(p)$.

$\exists\, \bar{k}$ s.t. $k \geq \bar{k} \Rightarrow$

$$|f(p_{n_k}) - f(q_{n_k})| \leq |f(p_{n_k}) - f(p)| + |f(p) - f(q_{n_k})|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Contradiction!

Connectedness

Def A metric space is <u>disconnected</u> if $\exists$ open subset U, V s.t.

$$U \neq \phi, \; V \neq \phi, \; U \cap V = \phi \;\; U \cup V = M.$$

This is called a <u>separation</u> of M. If $\nexists$ sep of M, we say M is <u>connected</u>.

A similar def is applied to subsets of a metric sp. in which case $S \subset U \cup V$.

~~Thm Thee cont.~~

Ex $[1,2] \cup \{17\}$ is ~~not~~ disconnected.

We will show intervals are connected.

Thm The cont. image of a conn'd sp is conn'd.

Pf See text. Very basic.

Thm If M is homeo to N, both are conn'd or neither are.

Thm Generalized IVT

Let $f: M \to \mathbb{R}$ be cont. Suppose M is connected and $a, b \in M$. If $f(a) < y < f(b)$, then $\exists\ c \in M$ s.t. $f(c) = y$.

Pf ~~Let~~ Suppose not. Let

$$U = \{x \in M \mid f(x) < y\}$$

$$V = \{x \in M \mid f(x) > y\}.$$

$a \in U$, $b \in V$. and clearly $U \cap V = \emptyset$.

If $x \in M \setminus (U \cup V)$, then $f(x) = y$, so $U \cup V = M$.

But, are U & V open? (Book skips this)

$$U = f^{-1}((-\infty, y)), \quad V = f^{-1}((y, \infty))$$

are open since f is cont. $\blacksquare$

Thm $\mathbb{R}$ is connected.

pf: Suppose U, V is a separation of $\mathbb{R}$. Then U is open, closed, not empty and $\neq \mathbb{R}$. But U is a countable union of disjoint open intervals, whose end pts ~~are~~ are not in U. U is closed, so it contains its end pts (why?). Therefore U has no end pts, ie $U = \mathbb{R}$. Contradiction. $\blacksquare$

Applications

Intervals are connected.

Pf: (a,b), $(-\infty, a)$, (a,∞) are homeo to $\mathbb{R}$.

$\exists$ a cont. function from $\mathbb{R}$ onto $[a,b]$, $[-\infty, a]$, $[a,\infty)$ $[a,b)$ and $(a,b]$. Example: Let

$$f(x) = \begin{cases} 1 & x \geq 1 \\ x & x \in [-1, 1) \\ -1 & x \leq -1 \end{cases}$$

Prove f is cont. $f(\mathbb{R}) = [-1, 1]$.

Let $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$ is connected.

Quasi-proof: The map $t \mapsto (\sin t, \cos t)$ is a cont. map from $\mathbb{R}$ onto S^1!

- $\mathbb{R}$ is not homeo to $\mathbb{R}^n$ $n>1$.

Pf: Suppose $h: \mathbb{R} \to \mathbb{R}^n$ is a homeo.

Let $h': \mathbb{R} \setminus \{0\} \to \mathbb{R}^n - \{h(0)\}$.

Check that h' is a homeo.

Clearly $\mathbb{R} \setminus \{0\}$ is not connected.

But $\mathbb{R}^n$ - a point is connected (I'll show this later)

- $[0,1]$ is not homeo to S^1.

- X is not homeo to Y.

Thm Let S be connected. Then $\bar{S}$ is connected.

Pf: Suppose $\bar{S}$ is disconnected.
$\exists\, U, V$ s.t. $\bar{S} = U \cup V$, $U \cap V = \emptyset$, $U \neq \emptyset$, $V \neq \emptyset$
and both are open.

Let $A = S \cap U$ and $B = S \cap V$.
If $A = \emptyset$, then U contains a limit point of S (since $U \neq \emptyset$)
and thus limit pt has a nbhd, U, that misses S. $\divideontimes$
Thus $A \neq \emptyset$. Likewise $B \neq \emptyset$.
Clearly $A \cap B = \emptyset$ and $A \cup B = S$.
~~Are they open?~~

~~Let $p \in A$. $\exists\, \varepsilon > 0$ s.t. $B(p, \varepsilon) \subset$~~

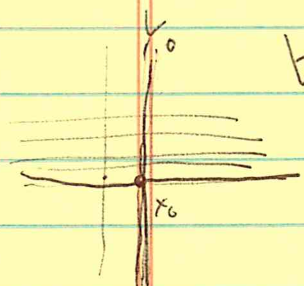
Ex The converse is false: $(0,1) \cup (1,2)$ is not conn'd
but its closure $[0,2]$ is.

Thm If A & B are connected, and $A \cap B \neq \emptyset$, then
$A \cup B$ is connected.

Pf See textbook. Generalizes to arbitrary unions.

Thm The product of two connected spaces is conn'd.

Pf Let X and Y be conn'd. Let $x_0 \in X$. Let $Y_0 \subset X \times Y$ be $\{x_0\} \times Y$. It is homeo to Y and thus conn'd.



$\forall y \in Y$ let $X_y = (X \times \{y\}) \cup Y_0$

Then $X \times Y = \bigcup_{y \in Y} X_y$ and $\bigcap X_y = Y_0 \neq \emptyset$.

Thus $X \times Y$ is conn'd.

Def Let $p, q \in M$, a metric sp. Let $f : [a,b] \to M$ be cont. with $f(a) = p$, $f(b) = q$. Then we say f is a path from p to q in M.

A metric sp M is <u>path conn'd</u> if $\forall p, q \in M$ $\exists$ a path from p to q in M.

Fact In $\mathbb{R}^n$ every open conn'd subset is path conn'd. This is import in calculus for defining path integrals.

Fact Path conn'd $\Rightarrow$ conn'd, but the converse is false.

Outline of proof

Let M be path conn'd but not conn'd.
Let U, V be a separation.
Let $p \in U$, $q \in V$. Let $f : [0,1] \to M$ be a path from p to q.

Show that $f^{-1}(U), f^{-1}(V)$ is a separation of $[0,1]$.

Next we outline why the converse is false.
Let $G = \{(x,y) \in \mathbb{R}^2 \mid y = \sin(\frac{1}{x}), 0 < x \leq \pi\}$.
Then one can show G is conn'd since it is the image of $(0, \pi]$ under

$$t \longmapsto (t, \sin(\tfrac{1}{t}))$$

and one can show this is continuous.

Let $Y = \{(0,y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}$.

One can show $\overline{G} = G \cup Y$. Thus $G \cup Y$ is conn'd.

But $G \cup Y$ is not path conn'd. See proof in Munkres Topology.

Coverings and Compactness

Let $K \subset M$

Let M be a metric space. Let $\mathcal{C}$ be a collection of subsets of M. ~~Let $K \subset M$.~~ s.t.

$$K \subset \bigcup_{C \in \mathcal{C}} C$$

Then we say $\mathcal{C}$ "covers" K. If $\mathcal{D} \subset \mathcal{C}$, that still covers K, then we say $\mathcal{D}$ is a subcover of $\mathcal{C}$. We will mainly be interested in open coverings.

Def If every open cover of K has a finite subcover, K is compact.

Ex $\{(\frac{1}{n}, 1) \mid n = 2, 3, \dots\}$ is an open covering of $(0,1)$ with no finite subcover.

Ex $\{(-n, n) \mid n = 1, 2, \dots\}$ is an open covering of $\mathbb{R}$ with no finite subcover.

Thm (54) For a subset A of a metric sp M
compact $\Leftrightarrow$ seq. compact.

Pf comp $\Rightarrow$ seq. comp is easy. The other direction require The Lebesgue Number Lemma.

($\Rightarrow$). Let $A \subset M$ be comp. and suppose it is not seq. comp. $\exists$ (a_n) in A, no subseq of which conv. in A. Therefore, $\forall a \in A$ $\exists$ $\varepsilon_a > 0$, s.t. $B(a, \varepsilon_a)$ ~~$\cap \{a_n\}$ is a finite set.~~ meets (a_n) at most finitely many times.

The collection $\mathcal{B} = \{B(a, \varepsilon_a) \mid a \in A\}$ is an open cover of A.

Let $\mathcal{B}' = \{B(a_i, \varepsilon_{a_i})\}_{i=1}^{m}$ be a finite subcover.

But each $B(a_i, \varepsilon_{a_i})$ meets (a_n) for at most finitely many n. But each $a_n \in \bigcup_{i=1}^{m} B(a_i, \varepsilon_{a_i})$.

Thus (a_n) has only finitely many distinct values. Thus $\exists$ $a = a_n$ for infinitely many n. But this forces a constant subseq, which of course converges. Contradiction!

Thus A is seq. comp. $\blacksquare$

Thm **Lebesgue Number lemma** For every open covering $\mathcal{U}$ of a seq. comp. set K, there is a real number $\lambda > 0$, s.t. $\forall a \in K, \exists U \in \mathcal{U}$ s.t. $B(a, \lambda) \subset U$.

Pf Suppose not. Let $\mathcal{U}$ be an open covering of a seq. comp. set K s.t. $\forall \lambda > 0, \exists a \in K$ s.t. $B(a, \lambda) \not\subset U \; \forall U \in \mathcal{U}$.

Let $\lambda_n = \frac{1}{n}$, $n = 1, 2, 3 \ldots$, and let $a_n \in K$ be such a pt. Since K is seq comp. $\exists$ a subseq (a_{n_p}) of (a_n) s.t. $\lim_{n \to \infty} a_{n_p} = a \in K$.

$\exists p_1 \in \mathbb{N}$ s.t. $\forall p \geq p_1$, $d(a_{n_p}, a) < r/2$.

$\exists p_2 \in \mathbb{N}$ s.t. $\forall p \geq p_2$ $\frac{1}{n_p} < r/2$. Suppose $p \geq \max(p_1, p_2)$.

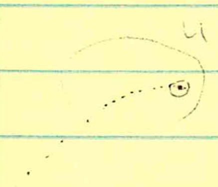
Claim $B(a_{n_p}, \frac{1}{n_p}) \subset B(a, r)$, which is in U.

Let $x \in B(a_{n_p}, \frac{1}{n_p})$. Then $d(a_{n_p}, x) \leq \frac{1}{n_p} < \frac{r}{2}$.

Thus $d(a, x) \leq d(a, a_{n_p}) + d(a_{n_p}, x) \leq \frac{r}{2} + \frac{r}{2} < r$.

Thus $x \in B(a, r)$. Then $B(a_{n_p}, \frac{1}{n_p}) \subset U$.

$\blacksquare$



Back to Thm 54. seq comp $\Rightarrow$ comp.

Let $\mathcal{U}$ be an open covering of a seq. comp. set K. We want to find a finite subset of $\mathcal{U}$ that covers K. Let λ be a Lebesgue number for $\mathcal{U}$.

Let $a_1 \in K$. $\exists\, U_1$ s.t. $B(a_1, \lambda) \subset U_1$.

If $K \subset U_1$, then $\{U_1\}$ is our finite subcover! Suppose not. Let $a_2 \in K \setminus U_1$. $\exists\, U_2 \in \mathcal{U}$ s.t. ~~$K \subset U_1 \cup U_2$~~ $B(a_2, \lambda) \subset U_2$. If $K \subset U_1 \cup U_2$, then $\{U_1, U_2\}$ is our subcover! Suppose not. Suppose there is no finite subcover. Then, we could continue this process and define ~~an~~ infinite seqs (a_n) and (U_n) with $B(a_n, \lambda) \subset U_n$, and $a_n \in K - \bigcup_{i=1}^{n-1} U_i$.

$\exists$ a subseq (a_{n_p}) that converges to some $a \in K$.

$\exists\, P_1 \in \mathbb{N}$, $\forall p \geq P_1$ $d(a_{n_p}, a) < \lambda$.

$\exists\, P_2 \in \mathbb{N}$, $\forall p \geq P_2$, $B(a_{n_p}, \lambda) \subset U_{n_p}$

~~But then the subseq cannot be Cauchy.~~

For $p \geq \max\{P_1, P_2\}$, $a \in U_{n_p}$.

But $a_{n_{p+1}}, a_{n_{p+2}}, \dots$ are not in U_{n_p}. So $a_{n_p} \not\to a$.

Contradiction! ■

Read about Total Boundedness and the Generalized Heine-Borel Thm on your own.

Perfect Metric Spaces

Def A metric space or subset is *perfect* if $M' = M$, that is, $\forall p \in M$, $\exists u_p$ s.t. $u_p \cap M$ = infinite, or every ~~ex~~ pt is a cluster pt.

Ex $[0,1]$ is perfect. $[0,1] \cup \{2\}$ is not.

Def A $p \in M$ is said to be *isolated* if $\exists r > 0$ s.t. $B(p,r) \cap M = \{p\}$. Perfect sets have no isolated points.

Thm Every non-empty, perfect, complete metric space is uncountable.

Pf Suppose not. Let M be $\neq \emptyset$, perfect, complete and countable. Since M is perfect every point is a cluster point, so M cannot be finite. Let

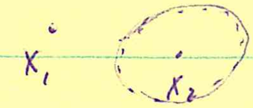
$$M = \{x_1, x_2, x_3, \dots \}$$

We are going to construct a nested seq of closed sets $C_1 \supset C_2 \supset C_3 \supset \cdots$, s.t. $x_i \notin C_i$.
We will find a pt $y \in \cap C_i$ and concluded $y \neq x_i$ $\forall i$, a contradiction.

Let $n_1 = 2$. So $x_{n_1} = x_2$.

Let $r_1 = \min\left\{1, \dfrac{d(x_1, x_{n_1})}{2}\right\}$.

Let $C_1 = \overline{B(x_{n_1}, r_1)}$



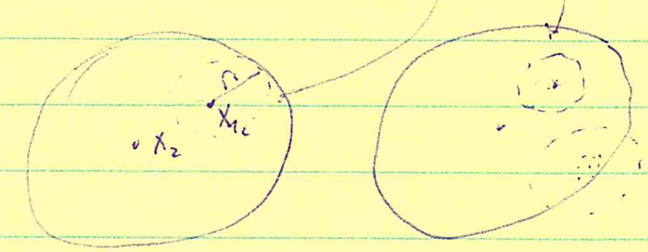
Then $x_1 \notin C_1$.

Since $\mathrm{int}(C_1)$ is an infinite set, $\exists\, n_2$, s.t. $x_{n_2} \in \mathrm{int}(C_1)$ but $x_{n_2} \neq x_2$.

Let $C_2 = \overline{B(x_{n_2}, r_2)}$ where

$$r_2 = \min\left\{\frac{1}{2}, \frac{d(x_{n_2}, x_2)}{2}, r_1 - d(x_{n_2}, x_{n_1})\right\}$$

Then $x_2 \notin C_2$ and $C_2 \subset C_1$



Since $\mathrm{int}(C_2)$ is infinite, $\exists\, x_{n_3} \in \mathrm{int}(C_2)$, $x_{n_3} \neq x_3$.

Let $C_3 = \overline{B(x_{n_3}, r_3)}$ where

$$r_3 = \min\left\{\frac{1}{3}, \frac{d(x_{n_3}, x_3)}{2}, r_2 - d(x_{n_2}, x_{n_3})\right\}.$$

Then $x_3 \notin C_3$, $C_3 \subset C_2$.

$x_k \notin C_k$, $C_k \subset C_{k-1}$.

Assuming $C_k = \overline{B(x_{n_k}, r_k)}$ has been defined,

$\exists\, x_{n_{k+1}} \in \operatorname{int} C_k$, s.t. $x_{n_{k+1}} \neq x_{k+1}$.

Define $C_{k+1} = \overline{B(x_{n_{k+1}}, r_{k+1})}$ where

$r_{k+1} = \min\left\{ \frac{1}{k+1}, \frac{d(x_{n_{k+1}}, x_{k+1})}{2}, r_k - d(x_{n_k}, x_{n_{k+1}}) \right\}$

Then $x_{k+1} \notin C_{k+1}$, and $C_{k+1} \subset C_k$.

~~Now we~~ Continuing inductively, [Munkres Thm 8.4]

We have $C_1 \supset C_2 \supset C_3 \supset \cdots C_k \supset C_{k+1} \supset \cdots$

with $x_k \notin C_k$. Further, each $C_k \cap \{x_1 \ldots x_k\} = \emptyset$.

Since radius of $C_k \leq \frac{1}{k}$, the seq of centers $\{x_{n_i}\}_{i=1}^{\infty}$, is Cauchy!

By the completness of M, $\lim_{i=\infty} x_{n_i}$ exists. Call it Ω.

Then $\Omega \in M$, but $\Omega \in C_k$ $\forall k$. Thus $\Omega \neq x_k$

for any k. Contradiction!

$\blacksquare$

Cantor Sets.

Def Let $C^0 = [0,1]$, $C^1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$, $C^2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$, etc.

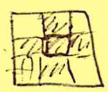
In general, let $C^{k+1} = C^k - \bigcup_{i=\text{odd}} \left(\frac{i}{3^{k+1}}, \frac{i+1}{3^{k+1}}\right)$.

Draw picture.

Let $C = \bigcap_{i=0}^{\infty} C^i$; it is called the Cantor set or the Cantor middle thirds set.

~~Thm~~ **Fact** It is a "fractal". Give informal definition of Hausdorff dimension. $c = m^d$, $2 = 3^d$, $d = \log_3 2 = \frac{\ln 2}{\ln 3} = 0.63929754\ldots$

Other examples. middle $\frac{1}{2}$; $2 = 4^d$ $d = \log_4 2 = 0.5$.



$m=3$ $c=5$ $5 = 3^d$ $d = \log_3 5 = \frac{\ln 5}{\ln 3} = 1.46497\ldots$

Fact Measure, i.e. length, is 0, a zero set, measure zero.

length of C^n is $(\frac{2}{3})^n$.

$\lim_{n \to \infty} (\frac{2}{3})^n = 0$.

Fact Removing middle $\frac{1}{4}$. Get Fat Cantor set.

C is a "zero set" or has "measure zero".

C_0	1	0
C_1	$\frac{2}{3}$	$\frac{1}{3}$
C_2	$\frac{4}{9}$	$\frac{1}{3}+\frac{2}{9}$
C_3	$\frac{8}{27}$	$\frac{1}{3}+\frac{2}{9}+\frac{4}{27}$
C_4	$\frac{16}{81}$	$\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\frac{8}{81}$
	$\vdots$	
C_n	$\frac{2^n}{3^{n+1}} = \frac{1}{3}\left(\frac{2}{3}\right)^n \to 0$	

$24-8=16$

$$\sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = \frac{1}{3}(3) = 1$$

A Fat Cantor Set.

F_0	1	0
F_1	$\frac{3}{4}$	$\frac{1}{4}$
F_2	$\frac{10}{16}$	$\frac{1}{4}+\frac{2}{16}$
F_3	$\frac{36}{64}$	$\frac{1}{4}+\frac{2}{16}+\frac{4}{64}+\frac{8}{256}$
F_4	$\frac{136}{256}$	
	$\downarrow$	

$\frac{40}{66}-4$

$\underline{36\times 4-8}$

$\frac{144-8}{256} = \frac{136}{256}$

$$\sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{2}{4}\right)^n = \frac{1}{4}\,\frac{1}{1-\frac{1}{2}} = \frac{1}{4}(2) = \frac{1}{2}.$$

Def A metric sp is totally disconnected if $\forall p \in M, r > 0$, $\exists$ clopen nbhd of p in $B(p,r)$.

Fact Each "piece" of C is clopen. Ex $C \cap [0, \frac{1}{3}]$ is closed but $C \cap [0, \frac{1}{3}] = C \cap (-\frac{1}{10}, \frac{1}{3} + \frac{1}{10})$ and so is open.

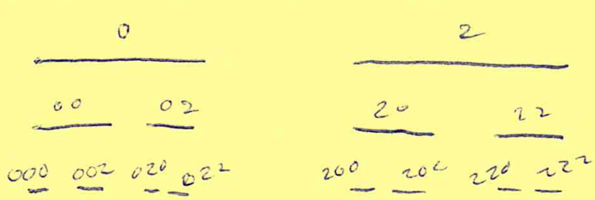
Thm C is compact, $\neq \emptyset$, perfect and totally disconnected.

Pf: See text. Testable!

Cor C is uncountable.

There is a more direct way to see this.

Address

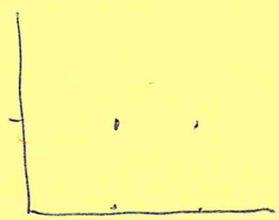


Take one connected interval from each C^n. There intersection is a unique point, $p \in C$. There is a one-to-one correspondence between point is C and such sequences of intervals, and with every infinite seq of 0's and 2's. In fact, this gives the base 3 representation of each pt in C. $C = x \in [0,1]$, s.t. x has a base 3 expansion s.t. the expansion has only 0's & 2's.

From this one can construct an onto map from C to $[0,1]$.
Map $(s_n) \to (a_n)$ where $a_n = 0$ if $s_n = 0$
and $a_n = 1$ if $s_n = 2$.
Then think of (a_n) as the base 2 representation of a number in $[0,1]$! It is onto, since every seq of 0's & 1's gets hit!
[Note: the set of infinite "words" from a finite alphabet is uncountable.]

This map is not one-to-one.

$.02222\ldots = .1 = \frac{1}{3}$ (base three) $\longmapsto .0111 = .1 = \frac{1}{2}$
$.2000\ldots = \frac{2}{3} \longmapsto .1 = \frac{1}{2}$



All the gaps are "healed".

~~It gets weirder~~:

~~Thm Let M be a compact, $\neq \emptyset$,~~ metric space.
~~$\exists$~~

Furthermore: this map is continuous!
Think about the inverse image of open intervals.

<u>Thm</u> Let $M \neq \emptyset$, be a comp. m. sp. $\exists$ a cont. surjection of C onto M.

<u>Idea of proof</u> For any k, cover M with balls of radius $\frac{1}{2^k}$ and then pass to a finite subcover. Let $\mathcal{M}_k$ = this finite cover.
Let $p \in M$. For each k, $\exists\, U_i \in \mathcal{M}_k$ with $p \in U_i$.
Think of these as a (non unique) address for p.

For $c \in C$ find a map from its address to a valid address for a point in M. Map c to this point.

This is onto. The fact that the diameters of the sets in $\mathcal{M}_k$ go to zero can be used to show this map is continuous. [Many details are needed.].

<u>Def</u> A Peano Curve is a continuous path in $\mathbb{R}^2$ which is space filling, the image in $\mathbb{R}^2$ has nonempty interior.

<u>Fact</u> Peano Curves exist.

Pf idea: Let B^2 = closed unit ball in $\mathbb{R}^2$.
$\exists$ cont. onto map $\sigma: C \to B^2$.
Define $\tau: [0,1] \to B^2$ by

$$\tau(x) = \begin{cases} \sigma(x) & x \in C \\ (1-t)\sigma(a) + t\sigma(b) & \text{if } x \text{ is in the } (a,b) \text{ gap} \\ & \text{and } x = (1-t)a + tb. \end{cases}$$

See text for cont. details. $\boxed{?}$

Def A metric space M is a Cantor space if it is comp. non-empty, perfect, & totally disconnected.

Thm (Moore-Kline) C is homeo to every Cantor space.

Idea of proof: The covers we used in Thm 65 can be made of disjoint pieces. ~~The~~ Now, address are unique and the cont. surjection become one-to-one. By Thm 42 (pg 81) the inverse is cont. ∎

Think about the following weird fact, $C \times C$ is a Cantor space, thus $C \cong C \times C$. (~~See~~ Exersices, 109, 112, 113, 114 ~~P~~. Could be a thesis!

↳ When does this happen. Open!

There is tons more one could say. ~~See~~ textbook.

We go on to Section 7.

Section 7 Completion

Thm (76) Every metric space can be completed. This means given M, $\exists$ a complete m. sp. $\hat{M}$ sat. $\exists\ h: M \to \hat{M}$ s.t. $\overline{h(M)} = \hat{M}$ and h is an isometry onto its image, i.e. $d(p,q) = \hat{d}(h(p), h(q))$. ↳ see #18.

pf Let $\bar{p} = (p_n)$, $\bar{q} = (q_n)$ be Cauchy seq's in M. Define $\bar{p} \sim \bar{q}$ if $d(p_n, q_n) \to 0$. It is easy to check that $\sim$ is an eq. relation. Let $\hat{M}$ = Cauchy seq's in M mod $\sim$.

We define a metric D on $\hat{M}$ by

$$D(P, Q) = \lim_{n\to\infty} d(p_n, q_n),$$

where $(p_n) \in P$ and $(q_n) \in Q$.

We need to check the following.

(a) D is well defined, meaning $D(P, Q)$ does not depend on which seq's are chosen, and

(a') it is a metric.

(b) Let $h: M \to \hat{M}$ be $h(p) = \bar{p} = (p, p, p, \dots)$. ~~We will~~ Then h is a isometry, onto its image.

(c) $\hat{M}$ is complete.

(d) $\overline{h(M)}$ is dense (see #10(a)).

(a) Fact (Lemma 77): $\forall p, q, x, y \in M$

$$|d(p,q) - d(x,y)| \le d(p,x) + d(q,y) \qquad (\star)$$

Pf: see book. Therefore,

$$|d(p_m, q_m) - d(p_n, q_n)| \le d(p_m, p_n) + d(q_m, q_n)$$

~~Let $L = \lim_{k\to\infty} d(p_k, q_k)$ ($\mathbb{R}$ is complete)~~

Therefore $d(p_n, q_n)$ is cauchy in $\mathbb{R}$.

Let $L = \lim_{n\to\infty} d(p_n, q_n) = D(P, Q)$ (This shows $D(P,Q)$ is defined!)

Suppose $(p_n') \in P$, $(q_n') \in Q$. $\left(\begin{array}{l} d(p_n', p_n) \to 0 \\ d(q_n', q_n) \to 0 \end{array} \right)$

Let $L' = \lim_{n\to\infty} d(p_n', q_n')$.

We claim $L = L'$. (Thus D is well defined.)

$$|L - L'| \le |L - d(p_n, q_n)| + |d(p_n, q_n) - d(p_n', q_n')| + |d(p_n', q_n') - L'|$$

The first and last terms go to zero.
The middle term, by $\star$, ~~...~~ is

$$\le d(p_n, p_n') + d(q_n, q_n').$$

These two terms go to zero. Thus $L = L'$.

Thus D is well defined.

(a) D is a metric

Since d is symmetric so is D:

$$D(P,Q) = \lim d(p_n, q_n) = \lim d(q_n, p_n) = D(Q,P).$$

$$D(P,P) = \lim d(p_n, p_n) = 0$$

If $D(P,Q) = 0$, then $\exists$ $(p_n) \in P$, $(q_n) \in Q$ s.t.

$\lim d(p_n, q_n) = 0$. But then $(p_n) \sim (q_n)$ and $P = Q$.

Finally, let $P, Q, R \in \hat{M}$.

$$D(P,Q) = \lim d(p_n, q_n) \le \lim d(p_n, r_n) + d(r_n, q_n)$$

$$= D(P,R) + D(R,Q).$$

(b) Let $h(p) = \bar{p} = (p, p, \cdots)$.

$$D(\bar{p}, \bar{q}) = \lim d(p,q) = d(p,q).$$

So h is an ~~is~~ isometry.

(c) $\hat{M}$ is complete. Let (P_k) be a Cauchy seq. in $\hat{M}$. We must show it converges in $\hat{M}$.

<u>Outline:</u> We define a seq (q_n) in M, show it is Cauchy. Then let $Q = [(q_n)]$ and show $P_k \to Q$.

$$\forall k \in \mathbb{N},\ \exists\ \bar{p}_k = (p_{k,n})_{n=1}^{\infty} \in P_k,\ \text{s.t. } \forall m,n \in \mathbb{N}$$

$$d(p_{k,m}, p_{k,n}) < \tfrac{1}{k}.$$

Pf: Pick any $\bar{x} \in P_k$. $\exists N$ s.t. $\forall m,n \geq N$ $d(x_m, x_n) < 1/k$. Delete the first N terms from $\bar{x}$ and call it $\bar{p}_k$.

Now, $\forall k$ choose $\bar{p}_k \in P_k$ with this property.

Let $q_n = p_{n,n}$. We will show (q_n) is cauchy.

Let $\varepsilon > 0$. $\exists N_0$ s.t. $k \geq N_0 \Rightarrow \frac{1}{k} < \varepsilon/3$.

$\exists N_1$ s.t. $k, \ell \geq N_1 \Rightarrow D(P_k, P_\ell) < \varepsilon/3$.

~~We know $d(p_{k,n}, p_{\ell,n}) \to d$ as $n \to \infty$~~

Now $\lim d(p_{k,n}, p_{\ell,n}) = D(P_k, P_\ell) < \varepsilon/3$

so $\exists N_2$ s.t. $m, n \geq N_2 \Rightarrow d(p_{k,n}, p_{\ell,n}) < \varepsilon/3$.

Let $N = \max\{N_0, N_1, N_2\}$. If $n, k, \ell \geq N$

$$d(q_k, q_\ell) = d(p_{kk}, p_{\ell\ell}) \leq d(p_{kk}, p_{kn}) + d(p_{k,n}, p_{\ell n}) + d(p_{\ell n}, p_{\ell\ell})$$

$$\leq \frac{1}{k} + \frac{\varepsilon}{3} + \frac{1}{\ell} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus (q_n) is Cauchy. Let $Q = [(q_n)] \in \hat{M}$. This is our putative limit.

Now we show $P_k \to Q$. Let $\varepsilon > 0$. $\exists N$ s.t. $k \geq N \Rightarrow \frac{1}{k} < \varepsilon/2$ and $k, n \geq N \Rightarrow d(q_k, q_n) < \varepsilon/2$.

Then $d(p_{k,n}, q_n) \leq d(p_{k,n}, p_{kk}) + d(\underset{= q_k}{p_{kk}}, q_n)$

$$< \frac{1}{k} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $P_k \to Q$.

(d) Exercise 101.

(a) $\overline{h(M)} = \hat{M}$.

(b) If $\hat{N}$ is another comp. m. sp. s.t. $\exists\, h' : M \to \hat{N}$ s.t. $\overline{h'(M)} = \hat{N}$,
isometry

Then $\exists$ ~~...~~ an isometry $K : \hat{M} \to \hat{N}$.

One can use this process to form a completion of $\mathbb{Q}$. But you also need to define the ordered field ops on $\hat{\mathbb{Q}}$.

Once this is done $\hat{\mathbb{Q}}$ is isometric to $\mathbb{R}$ and the isometry is an order preserving field isomorphism.

See textbook.