Chapter 3 Functions of a Real Variable

Section 1 §5 **Differentiation.** I assume you know basic facts about derivatives and how to prove them. You should get a calc text and compare and think about how you would teach calc.

Note: Most Calc I books prove $(x^n)' = nx^{n-1}$ using the binomial thm. This is bad. Instead do prod. rule first, then use induction.

Note: MVT, text says important for error estimates. See webpage for handout on this that I use as optional reading in my Calc II classes.

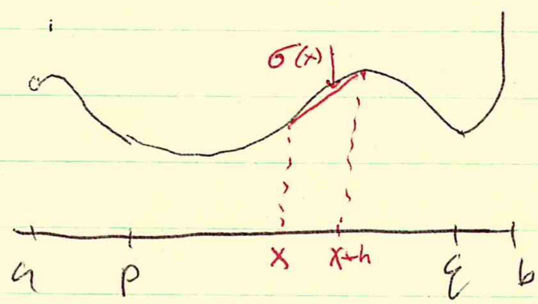
Here is something you might not know

Thm(7) If $f:(a,b)\to\mathbb{R}$ is diff. then f' has the intermediate value property; that is, if $f'(p) < \gamma < f'(q)$ $\exists\, r \in (p,q)$ or (q,p) s.t. $f'(r) = \gamma$.

Fact f' need not be cont. You saw this in the first hwk set (see my solutions and the Example on pg 146.)

Pf Suppose $a < p < q < b$ and $\alpha = f'(p) < \gamma < f'(q) = \beta$. (The other case, $q < p$, is similar.)

Choose $h \in (0, p-q)$ and consider the line segments $\sigma(x)$ from $(x, f(x))$ to $(x+h, f(x+h))$ as x ranges from p to $q-h$. Note $\sigma : [p, q-h] \to$ the set of line segments in $\mathbb{R}^2$.



Let $s(x) =$ slope of $\sigma(x)$. Thus $s(x) = \dfrac{f(x+h) - f(x)}{h}$, and thus is a cont. func. from $[p, q-h]$ to $\mathbb{R}$.

Let $0 < \varepsilon < \min\{|f'(p) - \gamma|, |f'(q) - \gamma|\}$.

$\exists\ \delta > 0$ s.t. $0 < h < \delta \Rightarrow |s(p) - f'(p)| < \varepsilon$ and $|s(q-h) - f'(q)| < \varepsilon$, since $\lim\limits_{h \to 0} s(x) = f'(x)$. Fix h.

Then $s(p) < \gamma < s(q-h)$. Since s is cont. $\exists\ x_0 \in (p, q-h)$ s.t. $s(x_0) = \gamma$.

Apply the MVT to f over (x_0, x_0+h), to get $\exists\ r \in (x_0, x_0+h)$ s.t. $f'(r) = \dfrac{f(x_0+h) - f(x_0)}{h} = s(x_0) = \gamma$. ∎

$C^0 = C^0(X, Y)$ = cont. functions from X to Y.
We assume X = some real interval and $Y = \mathbb{R}$.

$C^1 = \{ f: \mathbb{R} \to \mathbb{R} \mid f'$ exists and is continuous $\}$.

$C^2 = \{ \quad \mid f''$ exists and is cont. $\}$

etc.

$C^\infty = \{ \quad \mid$ ~~All do~~ $f^{(n)}$ exists and is cont $\forall n \in \mathbb{N} \}$

It is clear that $C^0 \supset C^1 \supset C^2 \supset \cdots \supset C^\infty$, and it is easy to show then inclusions are proper.

<u>Def</u> A function $f: (a,b) \to \mathbb{R}$ is <u>analytic</u> if

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \text{ for some } x_0 \in (a,b).$$

[For finite interval $x_0 = \frac{b-a}{2}$.]

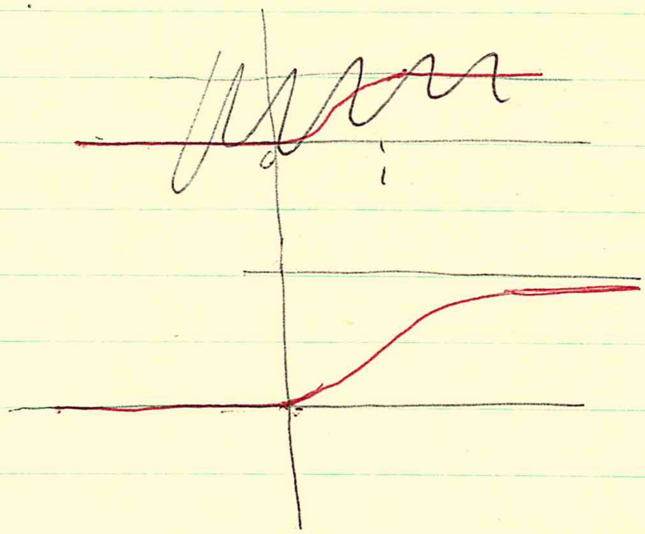
[The book's def. is "weaker", but implies this.]

C^ω = set of analytic functions.

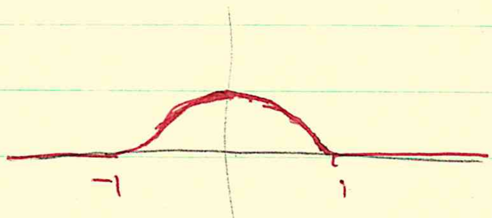
$C^\omega \subset C^\infty$. The inclusion is proper.

<u>pf</u> Let $e(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \le 0. \end{cases}$

In Exercise 14, you will show $e(x) \in C^\infty$, but not C^ω.



Also in 14, bump functions!



Taylor Polynomials and Approximation

Def Let $f:(a,b)\to\mathbb{R}$ be in C^r for some $r\geq 1$. ($r=\infty$ or ω)
~~[illegible]~~ Let $x_0\in(a,b)$. Then the r^{th} order Taylor poly of f centered at x_0 is

$$P_n(x) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

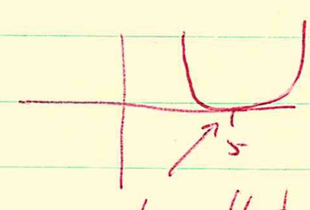
$$= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

Def A function $f:(a,b)\to\mathbb{R}$ is n^{th} order flat at $x=x_0$ if

$$\lim_{h\to 0}\frac{f(x_0+h)}{h^n} = 0$$

Ex Let $f(x)=(x-5)^4$. Then $f(x)$ is 3^{rd} order flat at $x=5$ since

$$\lim_{h\to 0}\frac{(5+h-5)^4}{h^3} = \lim_{h\to 0} h = 0.$$



how flat is it?

Ex Let $f(x) = (x+2)^5(x^4+1)$. Then $f(x)$ is 4^{th} order flat at $x=-2$ since

$$\lim \frac{(-2+h+2)^5(17)}{h^4} = \lim h = 0.$$

Thm Taylor Approx. Thm or T Remainder T.

Suppose $f:(a,b)\to\mathbb{R}$ is n^{th} order diff. at x_0. Then

(a) Let $R(x) = f(x) - P_n(x)$. Then R is n^{th} order flat at x_0.

(b) No other poly of degree $\le n$ has this property.

(c) If f is $n+1$ diff. on (a,b), then $\exists\ \theta$ in between x_0 and x s.t.

$$R(x) = \frac{f^{(n+1)}(\theta)}{(n+1)!}(x-x_0)^{n+1}.$$

~~Pf All derivatives of P exists, so R has n^{t} at least derivatives.~~

~~At $x=x_0$, $R(x_0) = f(x_0) - P(x_0) = f(x_0) - f(x_0) - 0 - \cdots 0 = 0$.~~

~~and $R'(x_0) = f'(x_0)(x-x_0) + \cdots + = 0$~~

~~$f'(x_0) - f'(x_0) - 0 - 0 = 0$~~

$x_0 < x$ ~~By MVT $\exists\ \theta_1 \in (x_0, x)$ s.t.~~

~~$R'(\theta_1) = \dfrac{R(x) - \overset{=0}{R(x_0)}}{x-x_0}$.~~

~~$\exists\ \theta_2 \in (x_0, \theta_1)$ $R''(\theta_2) = \dfrac{R'(\theta_1) - \overset{=0}{R'(x_0)}}{1}$~~

Pf All derivatives of P exist, so R has at least n derivatives.

At $x = x_0$, $R(x_0) = f(x_0) - \left[f(x_0) + f'(x_0)(x_0 - x_0) + \frac{f''(x_0)}{2}(x_0 - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)(x_0 - x_0)^n}{n!} \right]$

$= 0.$

$R'(x_0) = f'(x_0) - \left[f'(x_0) + f''(x_0)(x_0 - x_0) + \frac{f'''(x_0)}{2}(x_0 - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{(n-1)!}(x_0 - x_0)^{n-1} \right]$

$= 0$

$R''(x_0) = f''(x_0) - \left[f''(x_0) + f'''(x_0)(x_0 - x_0) + \cdots + \frac{f^{(n)}(x_0)}{(n-2)!}(x_0 - x_0)^{n-2} \right]$

$= 0$

$\vdots$

$R^{(n)}(x_0) = f^{(n)}(x_0) - f^{n}(x_0) = 0.$

Suppose $x_0 < x$, By the MVT $\exists\, \theta_1 \in (x_0, x)$ s.t.

$$R'(\theta_1) = \frac{R(x) - \cancel{R(x_0)}^{\,0}}{x - x_0}, \text{ thus } \Rightarrow R'(\theta_1)(x - x_0) = R(x).$$

By the MVT $\exists\, \theta_2 \in (x_0, \theta_1)$ s.t.

$$R''(\theta_2) = \frac{R'(\theta_1) - \cancel{R'(x_0)}^{\,0}}{\theta_1 - x_0}; \text{ thus } \Rightarrow R''(\theta_2)(\theta_1 - x_0) = R'(\theta_1)$$

or $R''(\theta_2)(\theta_1 - x_0)(x - x_0) = R(x)$.

By the MVT $\exists\, \theta_3 \in (x_0, \theta_2)$ s.t.

$$R'''(\theta_3) = \frac{R''(\theta_2) - \cancel{R''(x_0)}^{\,0}}{\theta_2 - x_0}; \text{ this implies}$$

$$R'''(\theta_3)(\theta_2 - x_0) = R''(\theta_2), \text{ so}$$

$$R'''(\theta_3)(\theta_2 - x_0)(\theta_1 - x_0)(x - x_0) = R(x).$$

$$\vdots$$

$$R^{(n-1)}(\theta_{n-1})(\theta_{n-2} - x_0)(\theta_{n-3} - x_0)\cdots(\theta_1 - x_0)(x - x_0) = R(x)$$

for $x_0 < \theta_{n-1} < \theta_{n-2} < \cdots < \theta_1 < x$.

Now $$\left|\frac{R(x)}{(x-x_0)^n}\right| = \left|\frac{R^{n-1}(\theta_{n-1})(\quad)\cdots(x-x_0)}{(x-x_0)^n}\right| \le \left|\frac{R^{(n-1)}(\theta_{n-1})}{x-x_0}\right|$$

$$\le \left|\frac{R^{(n-1)}(\theta_{n-1})}{\theta_{n-1} - x_0}\right| \longrightarrow |R^n(x_0)| = 0.$$

as $x \to x_0$ ($\theta_{n-1} \to x_0$) from the right.

The case $x_0 > x$ is similar.

For b & c see textbook. $\blacksquare$

Read about Inverse Functions & derivatives on your own.

Proof that e is irrational

Reference: Principles of Mathematical Analysis by Rudin, pages 48-50.

Theorem: e is irrational.

Proof: Suppose e is rational and that $e = p/q$, where $p > 0$ and $q > 0$. In fact we can assume $q > 1$ since e is not an integer. (You should prove this!) We will derive a **contradiction**, namely that there is an integer between 0 and 1!

We know from Taylor's theorem that $e = \sum_{k=0}^{\infty} \frac{1}{k!}$. Let $e_n = \sum_{k=0}^{n} \frac{1}{k!}$.

Then $e - e_n = \dfrac{1}{(n+1)!} + \dfrac{1}{(n+2)!} + \dfrac{1}{(n+3)!} + \cdots$

$$< \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots\right) =$$

$$\frac{1}{(n+1)!}\left(\frac{1}{1-\frac{1}{n+1}}\right) = \frac{1}{(n+1)!} \cdot \frac{n+1}{n+1-1} = \frac{1}{n!n}.$$

Thus, if we let $n = q$, we have $0 < e - e_q < \dfrac{1}{q!q}$.

Thus, $0 < q!(e - e_q) < \dfrac{1}{q} < 1$.

Now $q!e = \dfrac{q!p}{q} = (q-1)!p \in \mathbb{Z}$.

But also, $q!e_q = q!\left(1 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots + \dfrac{1}{q!}\right) \in \mathbb{Z}$.

Therefore, there exists an integer strictly between 0 and 1. Since this is absurd, we conclude that e cannot be expressed as a ratio of integers.

Section 2 Riemann Integration

Def A partition of $[a,b]$ is a finite set $P = \{x_0, x_1, \ldots x_n\}$ s.t. $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. A set of test pts or sample pts for a partition P is a set $T = \{t_1, t_2, \ldots, t_n\}$ s.t. $x_{i-1} \le t_i \le x_i$, $i = 1 \ldots n$. The mesh of a partition is $\max\limits_{i=1,\ldots,n} |\cancel{x_i - x_{i-1}}|$. Let $\Delta x_i = x_i - x_{i-1}$.

Def Let $f : [a,b] \to \mathbb{R}$. Let P be a partition of $[a,b]$ and let T be sample pts for P. Then the Riemann Sum of f wrt P and T is

$$R(f,P,T) = \sum_{i=1}^{n} f(t_i)\,\Delta x_i .$$

Def Let $f : [a,b] \to \mathbb{R}$. If $\exists\, I \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if P is any partition of $[a,b]$ with mesh $< \delta$, and for any T we have

$$|R(f,P,T) - I| < \varepsilon,$$

Then we say I is the Riemann integral of f over $[a,b]$ and write

$$I = \int_a^b f(x)\,dx.$$

When the integral exists we say f is Riemann integrable over $[a,b]$.

Q: Let $\mathcal{R}$ be the set of R.I. functions over $[a,b]$. What can we say about $\mathcal{R}$? Which functions are in $\mathcal{R}$? What structural properties does $\mathcal{R}$ have?

Thm (15): Every R.I. function over $[a,b]$ is bdd.

Outline of Pf: Suppose f is unbdd. Let $\varepsilon=1$. → but that I exists. Let $\delta>0$. Choose any partition with mesh $<\delta$. $\exists k$ s.t. f is unbdd on $[x_k, x_{k+1}]$. Show that there is a $t_k \in [x_k, x_{k+1}]$ s.t. using this sample pt

$$|R(f,P,T) - I| > 1.$$

Def: Let $S \subset \mathbb{R}$. The characteristic function of S is

$$\chi_S = \begin{cases} 1 & x \in S \\ 0 & x \notin S. \end{cases}$$

Fact: $\chi_{\mathbb{Q}} \notin \mathcal{R}$. Pf: Let $\varepsilon = \frac{1}{2}$. For any partition P we can choose rational pts ~~for~~ T and get

$R(\chi_{\mathbb{Q}}, P, T) = 1$ or irrational pts and get $R(\chi_{\mathbb{Q}}, P, T) = 0$.

Note $\chi_{\mathbb{Q}}$ is discontinuous at every pt.

Fact $\mathcal{R}$ is a vector space under addition of functions and scalar mult. by real numbers. The book's proof just involves showing

$$\int_a^b f+g \, dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

and

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

This can also be interpreted to mean that

$$f \longmapsto \int_a^b f(x)\,dx$$

is a linear transformation from the vector sp $\mathcal{R}$ to the v. sp. $\mathbb{R}$.

See textbook for proofs of these and other basic properties of integration.

Darboux Integrability...

is a useful tool for determining Riemann integrability.

Def Let $f:[a,b]\to[-M,M]$, so f is bdd. Let $P=\{x_0,x_1,\dots x_n\}$ be a partition of $[a,b]$. In each subinterval let

$m_i=\inf\{f(t): t\in[x_{i-1},x_i]\}$ and $M_i=\sup\{f(t): t\in[x_{i-1},x_i]\}$.

(Note: f need not have a min or max in $[x_{i-1},x_i]$ since it need not be cont. Ex, $f(x)=\begin{cases}0 & x\in\mathbb{Q}\\ x & x\notin\mathbb{Q}\end{cases}$. Then max f on $[1,2]$ d.n.e, but sup $= 2$.)

Define, $L(f,P)=\sum_{i=1}^{n} m_i\Delta x_i$ and $U(f,P)=\sum_{i=1}^{n} M_i\Delta x_i$.

Then for any T, $L(f,P)\le R(f,P,T)\le U(f,P)$.

Let $\underline{I}=\sup\{L(f,P): P\in\mathcal{P}\}$ and $\overline{I}=\inf\{U(f,P); P\in\mathcal{P}\}$,

where $\mathcal{P}$ = all partitions of $[a,b]$. If $\underline{I}=\overline{I}$, then

we say f is Darboux integrable and $\underline{I}=\overline{I}$ is the

Darboux integral of f over $[a,b]$.

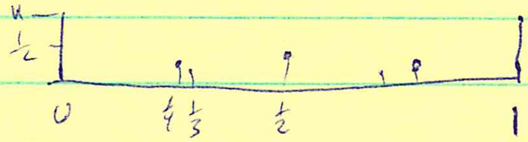
Thm(19) Riemann integrability and Darboux integrability are equivalent. If $\underline{I}=\overline{I}$, then $I=\underline{I}=\overline{I}$.

Before proving this we give ~~an~~ two applications.

Ex It is now immediate that $L(\chi_{\mathbb{Q}}, P) = 0$ and $U(\chi_{\mathbb{Q}}, P) = 1$ for all partitions. Thus $\chi_{\mathbb{Q}} \notin \mathcal{R}$.

Ex Let $f : [0,1] \to \mathbb{R}$ be $f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ \frac{1}{q} & x \in \mathbb{Q} \end{cases}$,

where $x = \frac{p}{q}$ is in reduced form. We take it that $f(0) = f(1) = 1$.



Rational Ruler function.

We claim f is Darboux integrable and hence by Thm 19 $f \in \mathcal{R}$, and that

$$\int_0^1 f(x)\,dx = 0.$$

Pf For any partition P it is clear that $L(f,P) = 0$, since each $m_i = 0$. Let $\varepsilon > 0$. We will find a partition P s.t. $U(f,P) < \varepsilon$. Then $\overline{I} = \underline{I} = 0$.

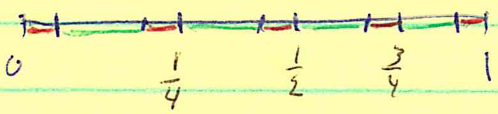
We will form a sequence of partitions (P_k) as follows. For any $k \geq 1$, there are exactly $k+1$ values of $x \in [0,1]$ where $f(x) \geq \frac{1}{k}$, namely $\frac{0}{k}, \frac{1}{k}, \frac{2}{k}, \ldots \frac{k}{k}$. We construct P_k by starting with

$$[0, \delta_k], \text{ and } \left[\tfrac{i}{k} - \delta_k, \tfrac{i}{k}\right], \; i = 1, \ldots K,$$

with $\delta_k > 0$ to be specified later.

On each of these $\sup f \le 1$. Cover the rest of $[0,1]$ with

$$[\delta_k, \tfrac{1}{k}],\ [\tfrac{i}{k}, \tfrac{i+1}{k} - \delta_k],\ i = 1, \dots n-1$$



$k = 4$

On each of these, $\sup f \le \frac{1}{k}$. The total length of these cannot exceed 1. Thus they contribute at most $1 \cdot \frac{1}{k}$ to the upper sum $U(f,P)$.

The others contribute at most $1 \cdot (k+1)\delta_k$. Thus

$$U(f, P_k) \le (k+1)\delta_k + \frac{1}{k}.$$

Let $\delta_k < \frac{1}{(k+1)^2}$ $\left(< \frac{1}{k}\right)$. Then, for large enough k,

$$U(f, P_k) \le \frac{1}{k+1} + \frac{1}{k} \le \frac{2}{k}.$$

Once $\frac{2}{k} < \varepsilon$ we have $U(f, P_k) < \varepsilon$.

Thus $\inf(U(f,P) \mid \text{all } P) < \varepsilon$, $\forall \varepsilon > 0$.

Thus $\overline{I} = 0$. ∎

To prove Thm (19) we need two definitions and some Lemmas.

Def If P is a partition of $[a,b]$ and P' is another partition of $[a,b]$, we say P' **refines** P if $P \subset P'$. Clearly $\text{mesh } P' \leq \text{mesh } P$.

Def Given two partitions P_1 and P_2, their **common refinement** is $P_1 \cup P_2$.

Lemma If P' refines P, then

$$L(f,P') \geq L(f,P) \text{ and } U(f,P') \leq U(f,P).$$

Pf Obvious.

Lemma For any two partitions, P_1, P_2, $L(f,P_1) \leq U(f,P_2)$.

Pf $L(f,P_1) \leq L(f,P_1 \cup P_2) \leq U(f,P_1 \cup P_2) \leq U(f,P_2)$.

Cor (*) $\underline{I} = \overline{I} \iff \forall \varepsilon > 0, \exists P$ s.t. $U(f,P) - L(f,P) < \varepsilon$.

Pf $\overline{I} = \underline{I} \iff \forall \varepsilon > 0 \ \exists P, P'$ s.t. $|U(f,P) - L(f,P')| < \varepsilon$.
Clearly, we can now drop the $|\;|$ signs.
If we pass to $P \cup P'$ as our partition the second lemma gives the result. $\blacksquare$

Pf of Thm 19 Darboux $\Longleftrightarrow$ Riemann.

Assume $f:[a,b]\to[-M,M]$ is Darboux int. over $[a,b]$. Then $\underline{I} = \overline{I}$; call this number I. We will show f is Riemann int. We are going to be working with three partitions of $[a,b]$

$$P_1 = \{y_0, y_1, y_2, \dots, y_{n_1}\}$$

$$P = \{x_0, x_1, \dots, x_n\}$$

and $P^* = P_1 \cup P$ and reindexed as $\{x_0^*, x_1^*, \dots, x_{n^*}^*\}$.

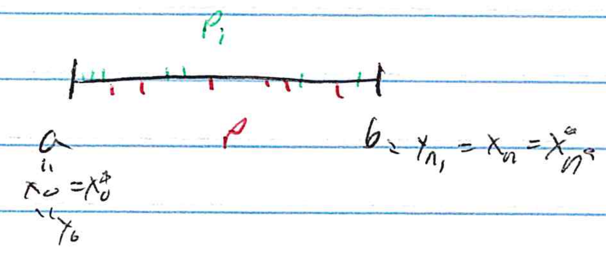
Let $\varepsilon > 0$. By $(\ast)$ assume P_1 is s.t. $U_1 - L_1 < \varepsilon/2$.

Let $\delta = \frac{\varepsilon}{8Mn_1}$. Let P be a partition with mesh $< \delta$.

Let $P^* = P \cup P_1$. By Lemma 1 $L_1 \le L^* \le U^* \le U_1$.

Thus $U^* - L^* < \frac{\varepsilon}{2}$.

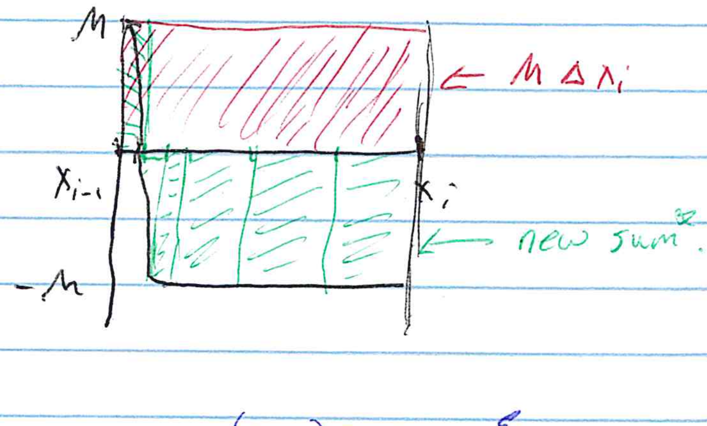
Now we compare the terms of $U = \sum M_i \Delta x_i$ and $U^* = \sum M_i^* \Delta x_i^*$.



If there is no x_j^* in (x_{i-1}, x_i) then the term $M_i \Delta x_i$ appear in both sums, but with possibly different indices. Since the end point of P_1^* and P are the same, there are at most $n_1 - 1$ members of $\{(x_{i-1}, x_i)\}$ that do contain an x_j^*.

For each of these $|M_i| \Delta x_i < M\delta$.
could be neg.

When we partition $[x_{i-1}, x_i]$ with any x_j^* in it what is the maximal impact of the sums? It is $2M\delta$. See figure:



Therefore,

$$u - u^* < (n_1 - 1) 2M\delta = (n_1 - 1) 2M \cdot \frac{\varepsilon}{8Mn_1} =$$

$$\left(\frac{n_1 - 1}{n_1}\right) \frac{\varepsilon}{4} < \frac{\varepsilon}{4}.$$

Similarly $L^* - L < \frac{\varepsilon}{4}$. Thus

$U - L = (U - U^*) + (U^* - L^*) + (L^* - L) < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.$

Since, $L \leq \underline{I} = I = \overline{I} \leq U$ and $L \leq R \leq U$.
$\hookrightarrow R(f, P^*, T)$
$\hookrightarrow P$

Therefore $|R - I| < \varepsilon$ and $f \in \mathcal{R}$.

Now assume $f \in \mathcal{R}$. Let $I = \int_a^b f(x)\,dx$.

We know f is bdd. Let $\varepsilon > 0$. $\exists\, \delta > 0$ s.t.

$\forall P, T$ with mesh $P < \delta$, $|R - I| < \varepsilon/4$.

Fix a P with mesh $< \delta$. In each $[x_{i-1}, x_i]$, $\exists\, t_i$ s.t.

$f(t_i) - m_i < \eta$. Let $T = \{t_1, \ldots, t_n\}$ be such choices.

Then $L(f, P) - R(f, P, T) = \sum m_i \Delta x_i - \sum f(t_i) \Delta x$

$= \sum (m_i - f(t_i)) \Delta x_i \leq \sum \eta \Delta x_i = \eta (b-a).$

Choose $\eta < \frac{\varepsilon}{4(b-a)}$. Now $R - L < \frac{\varepsilon}{4}$.

Similarly, $\exists\, T' = \{t_i'\}_{i=1}^{n}$ s.t. $U - R' < \frac{\varepsilon}{4}$.

Since mesh $P < \delta$ we have

$$U - L = (U - R') + (R' - I) + (I - R) + (R - L) < \varepsilon.$$

Thus $\forall \varepsilon > 0$, $\exists P$ s.t. $L < \underline{I} < I < \overline{I} < U < \varepsilon$.

Hence $\underline{I} = I = \overline{I}$. $\blacksquare$

It is now easy to prove continuous and piecewise cont. functions and in $\mathcal{R}$.
See text book for proofs. Know this.

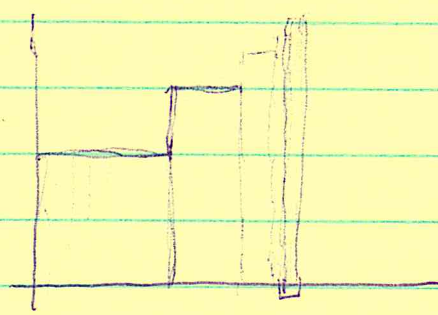
Ex Zeno's staircase function: $Z: [0,1] \to \mathbb{R}$ is

$$Z(x) = \begin{cases} \frac{1}{2} & x \in [0, \frac{1}{2}) \\ \frac{3}{4} & x \in [\frac{1}{2}, \frac{3}{4}) \\ \frac{7}{8} & x \in [\frac{3}{4}, \frac{7}{8}) \\ \frac{15}{16} & x \in [\frac{7}{8}, \frac{15}{16}) \\ \vdots \\ \frac{2^{k+1}-1}{2^{k+1}} & x \in [\frac{2^k-1}{2^k}, \frac{2^{k+1}-1}{2^{k+1}}) \\ \vdots \end{cases}$$

We claim $Z \in \mathcal{R}$ and $\int_0^1 Z(x)\,dx = \frac{2}{3}$.

$$\sum_{k=1}^{\infty} \frac{2^k - 1}{2^{2k}} = \sum_{k=1}^{\infty} \frac{1}{2^k} - \sum_{k=1}^{\infty} \frac{1}{4^k}$$

Pf



Let $P_k = \{0, \frac{1}{2^k}, \frac{2}{2^k}, \frac{3}{2^k} \cdots \frac{2^k}{2^k}\}$

For $i = 1, 2, \ldots 2^k - 1$, $m_i = M_i$.
So $U_k - L_k = (M_{2^k} - m_{2^k}) \frac{1}{2^k}$ $\quad M_{2^k} = 1$

$m_{2^k} = 1 - \frac{1}{2^k}$

So $U_k - L_k = \frac{1}{2^k} \frac{1}{2^k}$, can be made as small as we like. Hence $\underline{I} = \overline{I}$.

Thm(21) The Riemann-Lebesgue Thm.

A function $f: [a,b] \to \mathbb{R}$ is Riemann integrable iff it is bdd and its set of discontinuity pts is a zero set.

Def Recall, a set $Z \subset \mathbb{R}$ is a zero set (or has zero Lebesgue measure.) if for each $\varepsilon > 0$ $\exists$ a countable covering of open intervals $\{(a_i, b_i)\}$ s.t.

$$\sum_{i=1}^{\infty} b_i - a_i < \varepsilon.$$

could be finite

~~[illegible]~~

Ex finite sets, countable sets, the middle thirds Cantor set.

Any subset of a zero set is a zero set.

Any countable union of zero sets is a zero set.

Compare to def of Lebesgue outer measure on pg 363.

Let $A \subset \mathbb{R}$. The L outer me of A is

$$m^*(A) = \inf \left\{ \sum_{i=1}^{\infty, n} (b_i - a_i) \;\middle|\; \{(a_i, b_i)\}_{i=1}^{\infty, n} \text{ is a countable covering of } A \text{ by open intervals.} \right\}$$

$\boxed{\text{This is the heart of math 501}}$

Fact Let $Z_1, Z_2, Z_3, \ldots$ be a ct'able collection of zero sets. Then $Z = \cup Z_i$ is a zero set.

Pf Let $\varepsilon > 0$.

Cover Z_1 with open intervals $\{(a_{i1}, b_{i1})\}$ s.t.

$$\sum b_{i1} - a_{i1} \leq \frac{\varepsilon}{2}.$$

Cover Z_2 with open intervals $\{(a_{i2}, b_{i2})\}$ s.t.

$$\sum b_{i2} - a_{i2} \leq \frac{\varepsilon}{4}$$

$\vdots$

Cover Z_k with open interval $\{(a_{ik}, b_{ik})\}$ s.t.

$$\sum b_{ik} - a_{ik} \leq \frac{\varepsilon}{2^k}.$$

Now Z is covered by $\{(a_{ij}, b_{ij}) \mid i = 1, \ldots \alpha_j, j = 1, 2, \ldots\}$

{some α_j are finite, some are ∞.

The sum is

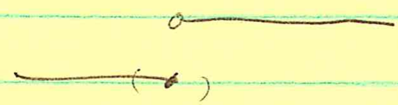
$$\sum_{j=1}^{\infty} \sum_{i=1}^{\alpha_j} b_{ij} - a_{ij} \leq \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon \left(\frac{1}{1-\frac{1}{2}} - 1 \right) = \varepsilon.$$

Thus Z is a zero set. $\blacksquare$

Def $\limsup_{t \to x} f(t) = \lim_{t \to x} \sup_{s \in (x-|t|, x+|t|)} f(s)$ ~~$(x+t, x+t)$~~

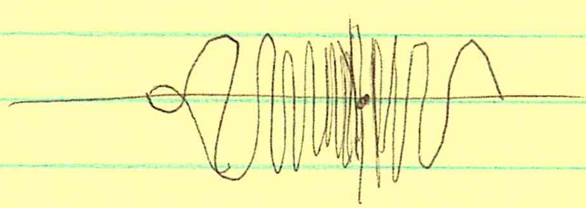
$\liminf_{t \to x} f(t) = \lim_{t \to x} \inf_{s \in (x-|t|, x+|t|)} f(s)$

Ex Let $f(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$.



$\limsup_{t \to 0} f(t) = 1$ $\liminf_{t \to 0} = 0$.

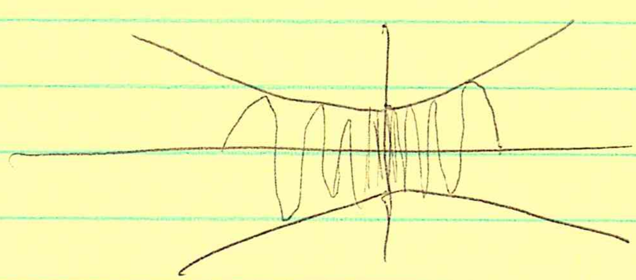
Ex Let $f(x) = \sin(\frac{1}{x})$ $x \ne 0$, 0 at $x = 0$.



$\limsup_{t \to 0} f(t) = 1$ $\liminf_{t \to 0} f(t) = -1$

Ex Let $f(x) = e^{|x|} \sin(\frac{1}{x})$ $x \ne 0$, 0 at $x = 0$.

Same.



Ex $f(x) = 0$ for $x \ne 0$, and $f(0) = 13.7$.

$\limsup_{t \to 0} f(t) = 13.7$ $\liminf f(t) = 0$

Def The oscillation of f at x is

$$\operatorname{osc}_x(f) = \limsup_{t \to x} f(t) - \liminf_{t \to x} f(t).$$

Ex use earlier ones.

Fact f is cont. at x iff $\operatorname{osc}_x(f) = 0$.

Now we are ready to prove the R-L Thm: $f : [a,b] \to \mathbb{R}$ is in $\mathcal{R}$ iff f is bdd and the set of discontinuous pts is a zero set.

pf Let D be the set of pts in $[a,b]$ where f is not cont. For each $k \in \mathbb{N}$ let

$$D_{\frac{1}{k}} = \{ x \in [a,b] \mid \operatorname{osc}_x(f) \geq \tfrac{1}{k} \}.$$

Then $D = \bigcup_{k=1}^{\infty} D_{\frac{1}{k}}$, a countable union.

Then D is a zero set iff each $D_{\frac{1}{k}}$ is.

Now, suppose $f \in \mathcal{R}$. Then it is bdd, so assume $f([a,b]) \subset [-M, M]$. Let $\varepsilon > 0$ and $k \in \mathbb{N}$. By Thm 19

$\exists$ a partition P s.t

$$U - L = \sum (M_i - m_i) \Delta x_i < \frac{\varepsilon}{k}.$$

If $[x_{i-1}, x_i]$ contains a pt of $D_{\frac{1}{k}}$ in its interior, then $M_i - m_i \geq \frac{1}{k}$. Call these intervals the type-I intervals of P.

$$\frac{1}{k} \sum_{\text{type I}} \Delta x_i \leq \sum_{\text{type I}} (M_i - m_i) \Delta x_i \leq \sum_{\text{all}} (M_i - m_i) \Delta x_i < \frac{\varepsilon}{k}.$$

Thus $\sum_{\text{type I}} \Delta x_i < \varepsilon$, that is the total length of these type I intervals is smaller than ε.

Then $D_{\frac{1}{k}}$, except for a finite number of pts is covered by $\{(x_{i-1}, x_i)\}_{\text{Type I}}$ with total length $< \varepsilon$. But $\varepsilon > 0$ was independent of k.

Thus $D_{\frac{1}{k}}$ is a zero set, and this holds for each $D_{\frac{1}{k}}$. Then D is a zero set.

Now for the converse. Assume D is a zero set. and again suppose $f([a,b]) \subset [-M, M]$. Let $\varepsilon > 0$.

~~By ...~~ Choose $k \in \mathbb{N}$ so that

$$\frac{1}{k} < \frac{\varepsilon}{2(b-a)}.$$

Clearly $D_{\frac{1}{k}} \subset D$ is zero set. $\exists$ ct'able open covering of $D_{\frac{1}{k}}$, (J_i), $J_i = (a_i, b_i)$ with total Length $\leq \frac{\varepsilon}{4M}$.

For each $x \in [a,b] \setminus D_{\frac{1}{k}}$, $\exists$ an open interval I_x containing x s.t. $\sup\{f(t) \mid t \in I_x\} - \inf\{f(t) \mid t \in I_x\} < \frac{1}{k}$.

Let $\mathcal{U} = \{J_i\}_{i=1}^{\infty \text{ might be finite}} \cup \{I_x \mid x \in [a,b] \setminus D_{\frac{1}{k}}\}$.

It is an open cover of $[a,b]$.

Let $\lambda > 0$ be its Lebesgue number!

Let $P = \{x_0 \dots x_n\}$ be any partition of $[a,b]$ with mesh $P < \lambda$.

We claim $U(f,P) - L(f,P) < \varepsilon$.

Each $[x_{i-1}, x_i]$ is $\subset$ a J_i or an I_x.

Let $\mathbf{J} = \{ i \in [1, \dots n] \mid [x_{i-1}, x_i] \subset J_q, \text{ for some } q \}$.

$\mathbf{J}$ is a list of subinterval of P where $osc_x(f)$

is $\geq \frac{1}{k}$. (big jumps).

Let $m = \max J$ (could be n).

$$U - L = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i = \sum_{i \in J} (M_i - m_i)\Delta x_i + \sum_{i \notin J} (M_i - m_i)\Delta x_i$$

$$\leq \sum_{i \in J} 2M \Delta x_i + \sum_{i \notin J} \tfrac{1}{k} \Delta x_i$$

$$\leq 2M \sum_{i=1}^{m} \Delta x_i + \frac{b-a}{k} < 2M\left(\frac{\varepsilon}{4M}\right) + \left(\frac{\varepsilon}{2(b-a)}\right)(b-a)$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $f \in \mathcal{R}$. $\blacksquare$

Implications

22 $C^0 \subset \mathcal{R}$. Piecewise cont. bdd $\subset \mathcal{R}$.

23 Let $S \subset [a,b]$. Then $\chi_S \in \mathcal{R}$ iff ∂S has measure zero. ($\partial \mathbb{Q} = \mathbb{R}$.)

24. Every monotone function is Riemann integrable.

Pf $\forall x \in [a,b]$ $f(a) \le f(x) \le f(b)$ or $f(b) \le f(x) \le f(a)$, so f is bdd.

Since f is bdd the number of pts $x \in [a,b]$ where $osc_x(f) \geq 1$ is finite.

Since f is bdd, the number of pt $x \in [a,b]$ where $osc_x(f) \geq \frac{1}{2}$ is finite

and so on.

The set of pt $x \in [a,b]$ where $osc_x(f) \neq 0$ is a countable union of finite ~~many~~ sets. Hence it is countable.

$\boxtimes$

25. If $f, g \in \mathcal{R}$, $f \cdot g \in \mathcal{R}$.

Pf: $D(f \cdot g) = D(f) \cup D(g)$

26. Let $f : [a,b] \to [c,d] \in \mathcal{R}$ and $\phi \in C^0([c,d])$

Then $\phi \circ f \in \mathcal{R}$. Pf: easy. But $f \circ \phi$ need not be R.I.

See Exercise 36.

27 $\quad f \in \mathcal{R} \Rightarrow |f| \in \mathcal{R}$

$a<b<c$

28 $\quad \int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

$\quad \hookrightarrow f \in \mathcal{R}([a,c])$.

29 Suppose $f : [a,b] \to [0, M] \in \mathcal{R}$ and $\int_a^b f(x)\,dx = 0$.
Then $f(x) = 0$ a.e.

Pf Suppose not. Let f be cont at x_0 with $f(x_0) > 0$.
$\exists\, \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow$ ~~$f(x) > f(x_0)$~~$/2$.

$$|f(x) - f(x_0)| < \frac{f(x_0)}{2}$$

$$-\frac{f_0}{2} < f - f_0 < \frac{f(x_0)}{2}$$

f

$$-\frac{f(x_0)}{2} < f(x) - f(x_0) < \frac{f(x_0)}{2}$$

$$\boxed{\frac{f(x_0)}{2} < f(x)}$$

Let $$g(x) = \begin{cases} \dfrac{f(x_0)}{2} & x \in (x_0 - \delta, x + \delta) \\ 0 & \text{otherwise.} \end{cases}$$

Then $0 \le g(x) \le f(x)$. Thus

$$0 \le \int_a^b g(x)\,dx \le \int_a^b f(x)\,dx = 0$$

$$0 \le \frac{f(x_0)}{2}(2\delta) \le 0$$

$$\Rightarrow f(x_0) = 0.$$

Def A function $f : \cancel{\mathbb{R}}\ [a,b] \to \mathbb{R}$ ~~satisfies~~ satisfies a Lipschitz condition if $\exists\, K \geq 0$

$$| f(s) - f(t) | \leq K | s - t |.$$

(If f is diff. that K is a bdd on $|f'|$.
If f is cont. it is uniformly cont.
In fact f is C^0, and is diff a.e.)

30 Let $f : [a,b] \to \mathbb{R} \in \mathcal{R}$.
Let $[c,d] = f([a,b])$. We are assuming $f \neq$ const as given.
Let $\psi : [c,d] \to [a,b]$ be one-to-one, onto, increasing.
($\psi(c) = a$, $\psi(d) = b$).
and suppose ψ^{-1} is Lipschitz.

Then $f \circ \psi : [c,d] \to \mathbb{R}$ is $\mathcal{R}$.

pf It follows that ψ (or ψ^{-1}) is a homeo.

Let D be the set of discont. pts of f.
Then
$D' = \psi^{-1}(D)$ is the set of discont pt of $f \circ \psi$.

Let $\varepsilon > 0$. Let K be ~~the~~ a Lip const for ψ^{-1}.

$\exists$ an open covering of D by intervals (a_i, b_i) with

$$\sum b_i - a_i \le \frac{\varepsilon}{K}.$$

Let $a_i' = \psi^{-1}(a_i)$ $b_i' = \psi^{-1}(b_i)$ $i = 1, 2 \ldots$

Then $\{(a_i', b_i')\}_{i=1}^{\infty}$ is an open cover of D' and

$$\sum b_i' - a_i' \le \sum K(b_i - a_i) \le \varepsilon.$$

Thus D' is a zero set and $f \circ \psi \in \mathcal{R}$. ∎

31. If $f : [a,b] \to [c,d] \in \mathcal{R}$ and $\psi : [c,d] \to [a,b]$ is a C^1 diffeo, then $f \circ \psi \in \mathcal{R}$.

Pf: Minor variation on 30.

32 FTC. Let $f:[a,b] \to \mathbb{R}$ be Riemann integrable. Define its indefinite integral as

$$F(x) = \int_a^x f(t)\,dt.$$

Then F is continous.

~~Thm~~ If x is a pt where f is cont, $F'(x) = f(x)$.

Pf We know f is bdd, so assume $|f(x)| < M$ over $[a,b]$.

$$|F(y) - F(x)| = \left| \int_a^y f(t)\,dt - \int_a^x f(t)\,dt \right| =$$

$$\left| \int_x^y f(t)\,dt \right| \leq M|x-y|.$$

Let $\varepsilon > 0$. Let $\delta = \frac{\varepsilon}{M+1}$. Then $|x-y| < \delta \Rightarrow M|x-y| < \varepsilon$.
Thus $F(x)$ is continuous on $[a,b]$.

Now assume f is cont. at $x \in [a,b]$.

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt.$$

Fix h for the moment. Let

$$m(x,h) = \inf\{f(s) : |s-x| \leq |h|\}, \text{ and}$$
$$M(x,h) = \sup\{f(s) : |s-x| \leq |h|\}.$$

Then

$$m(x,h) = \frac{1}{h}\int_x^{x+h} m(x,h)\,dt \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le \frac{1}{h}\int_x^{x+h} M(x,h)\,dt$$

Since f is cont. at x, you can show that $= M(x,h)$.

$$\lim_{h\to 0} m(x,h) = f(x) \quad \text{and} \quad \lim_{h\to 0} M(x,h) = f(x).$$

Therefore,

$$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x)$$

by the Squeeze Thm. ∎

Cor 33 The derivative of an indefinite Riemann integral exists almost everywhere and equals the integrand almost everywhere.

Cor 34 Every cont. func. $f:[a,b]\to\mathbb{R}$ has an anti-derivative, that $\exists\, F:[a,b]\to\mathbb{R}$, s.t. $F'(x) = f(x)$ always.

But in general, anti-der's may not exist. Let $g = \begin{cases} 1 & x \ge 0 \\ 0 & x < 0 \end{cases}$.

$\nexists\, G(x)$ s.t. $G'(x) = g(x)$ everywhere.

Pf: Suppose $G'(x) = g(x)$. Then show $G(x) = \begin{cases} c & x < 0 \\ x + c & x \ge 0. \end{cases}$

But then $G'(0)$ dne.

35 The Antiderivative Thm. Suppose $f:[a,b]\to\mathbb{R}$ is R. int. and that $G:[a,b]\to\mathbb{R}$ is an antiderivative of $f(x)$, that is $G'(x)=f(x)$ (always). Then

$$G(x)=\int_a^x f(t)\,dt + C.$$

Ex Calculus books give formulas that $\int \frac{1}{x}\,dx = \ln|x|+C$. But you have to be careful in interpreting these. This one is valid on $(-\infty,0)$ or $(0,\infty)$, but not both. Ex

Let $f(x)=\begin{cases}\ln x + 3 & x>0\\ \ln|x|+22 & x<0\end{cases}$

Then $f'(x)=\frac{1}{x}$.

Pf See textbook.

Devil's staircase. We will define a cont. func. $H:[0,1]\to\mathbb{R}$ s.t. $H'=0$ a.e. but H is not constant. It is based on Cantor's middle thirds set.

$$H(x)=\tfrac{1}{2} \qquad x\in[\tfrac{1}{3},\tfrac{2}{3}].$$

$$H(x)=\tfrac{1}{4} \qquad x\in[\tfrac{1}{9},\tfrac{2}{9}]$$
$$H(x)=3/4 \qquad x\in[\tfrac{7}{9},\tfrac{8}{9}]$$

$H(x) = \frac{1}{8}$ on $[\frac{1}{27}, \frac{2}{27}]$

$H(x) = 3/8$ on $[\frac{7}{27}, \frac{8}{27}]$

$H(x) = 5/8$ $[\frac{19}{27}, \frac{20}{27}]$

$H(x) = 7/8$ $[\frac{25}{27}, \frac{26}{27}]$

etc.

Thus $H'(x) = 0$ except on C. But C has measure zero.
It takes a little more work to show H is cont. on C.

We claim $H(x)$ has the following formula.

Let $x = (.x_1 x_2 x_3 \cdots)_3$ and $y = (.y_1 y_2, \cdots)_2$, then

$y = H(x)$ where

$$y_i = \begin{cases} 0 & \text{if } \exists\, k < i \text{ s.t. } x_k = 1 \\ 1 & \text{if } x_i = 1 \text{ and } \nexists\, k < i \text{ s.t. } x_k = 1 \\ \frac{x_i}{2} & \text{if } x_i = 0 \text{ or } 2 \text{ and } \nexists\, k < i \text{ s.t. } x_k = 1. \end{cases}$$

Translation,

Once $x_j = 1$, all $y_{j>i} = 0$.
The first time $x_j = 1$, $y_i = 0$.
if we have hit an $x_i = 1$ yet, $y_i = 0$ if $x_i = 0$
and $y_i = 1$ if $x_i = 2$.

For example, suppose $x \in [\frac{1}{3}, \frac{2}{3}]$.
Then $x = (.1 x_2 x_3 \cdots)$. Thus $y = (.100\cdots) = \frac{1}{2}$
If $x = (.2000 \quad)$ Then $y = (.100\cdots) = \frac{1}{2}$.

Suppose $x \in \left[\frac{\cancel{18}\,19}{27}, \frac{\cancel{8}\,20}{27}\right]$.

$$\frac{19}{27} = \frac{9}{27} + \frac{9}{27} + \frac{1}{27} = \frac{2}{3} + \frac{0}{9} + \frac{1}{27}$$

$$= (.201000\ldots)$$

$$\frac{20}{27} = .202000 = (.202000000)$$

$y_1 = 1$, $y_2 = 0$, $y_3 = 1$, $y_{i>3} = 0$

$$\frac{1}{2} + \frac{0}{4} + \frac{1}{8} = \frac{5}{8}.$$

<u>Extra Credit</u> Write a script in Maple or some other CAS to plot $H(x)$.

In general $H(x)$ is well defined. Suppose

$$x = (.x_1 x_2 \ldots x_\ell 222\ldots) = \underbrace{(.x_1 x_2 \ldots (x_\ell+1)\, 000\ldots)}_{=x'}$$

x_ℓ not 2.

$H(x) = ?$ $\qquad H(x') = ?$

if any of $x_1 \ldots x_{\ell-1}$ is one, $H(x) = H(x')$

if $x_\ell = 0$

$$H(x) = \left(\frac{x_1}{2}, \frac{x_2}{2} \ldots \frac{x_{\ell-1}}{2}, 0, 1, 1, 1 \ldots\right)_2 = \text{in base } 2.$$

$$H(x') = (\qquad 1\,0000 \qquad)_2$$

if $x_\ell = 1$,

$$H(x) = (\qquad , 1, 000 \qquad)$$

$$H'(x) = (\qquad , 1, 000 \ldots$$

✓

Now let $\varepsilon > 0$. $\exists k$ s.t. $\frac{1}{2^k} < \varepsilon$.
Let $\delta = 1/3^k$. Then

$$|x - x'| < \frac{1}{3^k} \Rightarrow \text{the first } k \text{ symbol is base 3 expansion are equal.}$$

$\Rightarrow$ The first k of $H(x)$ and $H(x')$ agree.
Hence

$$|H(x) - H(x')| \leq \frac{1}{2^k} < \varepsilon.$$ ∎

Def $\ln x = \int_1^x \frac{1}{t}\,dt$. for $x>0$. By the FTC $(\ln x)' = \frac{1}{x}$.

Further all derivatives of $\ln x$ exist, $\ln x \in C^\infty$

Claim: $\ln xy = \ln x + \ln y$.

pf: $\frac{d \ln(xy)}{dx} = \frac{1}{xy}\, y = \frac{1}{x}$.

$$\frac{d}{dx}(\ln x + \ln y) = \frac{1}{x}$$

Thus, $\ln xy - \ln x - \ln y = C$, a constant.

Let $x=y=1$. $\ln 1 - \ln 1 - \ln 1 = 0 = C$. ∎

Other familiar properties can by shown similarly.

Claim: $\ln x$ is one-to-one.

pf: Suppose $x<y$ and $\ln x = \ln y$. Then

$$\int_1^x \frac{1}{t}\,dt = \int_1^y \frac{1}{t}\,dt.$$

Then $\int_x^y \frac{1}{t}\,dt = 0$. But $\frac{1}{t} > 0$.

In fact

$$\int_x^y \frac{1}{t}\,dt \geq (y-x)\underbrace{\frac{1}{y}}_{\text{inf of } \frac{1}{t}} > 0.$$

∎

It is not hard to show $\exists\, x \in \mathbb{R}$ s.t. $\ln x > 1$. Clearly $\ln 1 = 0$. Let $e \in \mathbb{R}$ be the unique number s.t. $\ln e = 1$. Then e^x is the inverse of $\ln x$: $\ln e^x = x \ln e = x \cdot 1 = x$.

You can use Thm 13 (pg 152) to show $(e^x)' = e^x$. Hence $e^x \in C^\infty$.

You can check the n^{th} Taylor poly of e^x, centered at $x=0$, is

$$E_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

The Taylor series is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. It is not hard to show it converges $\forall\, x \in \mathbb{R}$. Thus

$$e = e^1 = \sum_{n=0}^{\infty} \frac{1}{n!}.$$ From this, we will show that e is irrational.

From Principles of Mathematical Analysis
Rudin, 48-50.

Thm e is irrational.

pf Suppose $e = p/q$, $p > 0$, $q > 0$.
Let $e_n = \sum_{k=0}^{n} \frac{1}{k!}$. Then

$$e - e_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots$$

$$< \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right) =$$

$$\frac{1}{(n+1)!}\left(\frac{1}{1 - \frac{1}{n+1}}\right) = \frac{1}{(n+1)!}\,\frac{n+1}{n+1-1} = \frac{1}{n!\,n}$$

Thus $0 < e - e_n < \frac{1}{n!\,n}$.

Let $n = q$. $0 < (e - e_q) < \frac{1}{q!\,q}$

$$0 < q!\,(e - e_q) < \frac{1}{q}$$

Since $e = \frac{p}{q}$, $q!\,e \in \mathbb{Z}$.

Also $q!\,(e_q) = q!\left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{q!}\right) \in \mathbb{Z}$.

Thus $q!\,(e - e_q) \in \mathbb{Z}$.

Since $q \geq 1$, $\frac{1}{q} < 1$. Then $\exists$ an integer

between 0 and 1. $\blacksquare$

The Riemann-Stieltjes Integral
(a brief introduction)

For more on this topic see, The Elements of Real Analysis, 2nd ed., by Robert Bartle, pgs 212-259.

Def Let $f, g : [a,b] \to \mathbb{R}$ be bdd. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$, and let $T = \{t_1, t_2, \ldots t_n\}$ be sample pts. The Riemann-Stieltjes sum of f wrt g is

$$S(P,T;f,g) = \sum_{k=1}^{n} f(t_k)\big(g(x_k) - g(x_{k-1})\big).$$

Think of g as a "charge" or "mass" along $[a,b]$, but it is not a density.

Def If $I \in \mathbb{R}$ is s.t. $\forall \varepsilon > 0$, $\exists P_\varepsilon'$, s.t. $\forall$ refinement P and sample set T, we have

$$|S(P,T;f,g) - I| < \varepsilon,$$

Then we say f is R.S. integrable wrt to g and write

$$I = \int_a^b f \, dg.$$

Ex Let $g(x) = \begin{cases} 0 & a \le x \le c, \\ 1 & c < x \le b, \end{cases}$ and suppose f is cont from the right at c. Then

$$\int_a^b f\,dg = f(c).$$

Pf. Look at partition member containing c.

Facts

$$\int_a^b \alpha_1 f_1 + \alpha_2 f_2 \, dg = \alpha_1 \int_a^b f_1\, dg + \alpha_2 \int_a^b f_2\, dg$$

$$\int_a^b f \, d(\alpha_1 g_1 + \alpha_2 g_2) = \alpha_1 \int_a^b f\, dg_1 + \alpha_2 \int_a^b f\, dg_2$$

If $g \in C^1([a,b])$ and f is RS int. wrt g, then

$$\int_a^b f\, dg = \int_a^b f g'\, dx$$

$$\int_a^b f\, dg + \int_a^b g\, df = f(x)g(x)\Big|_a^b.$$

Def Regard $C^0([a,b])$ as a vector space and give the sup norm, $\|f\| = \sup_{x \in [a,b]} |f(x)|$.

A <u>linear functional</u> is any linear function from $C^0([a,b]) \to \mathbb{R}$.

A linear functional G is <u>positive</u> if $f(x) \ge 0 \Rightarrow G(f) \ge 0$.

A linear functional G is <u>bounded</u> if $\exists\, M \ge 0$ s.t. $|G(f)| \le M\|f\|$.

<u>Fact</u> If g is monotone increasing ~~then~~ then $G(f) = \int_a^b f\,dg$ is a bdd. pos. lin. func.

<u>Thm</u> (Riesz Repersentation Thm) If $G : C^0([a,b]) \to \mathbb{R}$ is a bdd pos lin. functional, then $\exists$ a monotone inc. func $g : [a,b] \to \mathbb{R}$ s.t.

$$G(f) = \int_a^b f\,dg.$$

Section 3

Unlike baseball, a series is different from a sequence.

$$\sum_{n=1}^{\infty} a_n \text{ converges if } \lim_{k\to\infty} \sum_{n=1}^{k} a_n \text{ exists.}$$

There are many tests for determining convergence.

The Comparison Test ($\neq 0$) is the most obvious and the proof is a few lines.

Similar is the Integral Test. From this the p-series test follows: $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges iff $p > 1$.

Of course ~~you~~ we already covered the geom. series test. From this ~~the~~ Ratio and Root Tests follow.

Ratio Test Let $\sum a_k$ be an infinite series, $a_k \neq 0$.

Let
$$\lambda = \liminf_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| \qquad \rho = \limsup_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$$

If ρ is < 1, then $\sum a_n$ and $\sum |a_n|$ converge.

If $\lambda > 1$, then $\sum a_n$ diverges.

pf: Suppose $\rho < 1$. Choose $\beta \in (\rho, 1)$. $\exists N$ s.t. $n \geq N \Rightarrow$

$\cancel{k \geq K \Rightarrow |a_{k+1}/a_k| < \beta.}$ ~~Then~~

$\cancel{|a_{k+1}| < \beta|a_k| < \beta^2|a_{k-1}| < \beta^3|a_{k-2}| < \cdots < \beta^{k-K}|a_K|.}$

~~Thus~~ $\cancel{|a_k| < \beta^{k-K}|a_K|}$

$$\left|\frac{a_{n+1}}{a_n}\right| < \beta.$$

Then $|a_{n+1}| < \beta|a_n| < \beta^2|a_{n-1}| < \beta^3|a_{n-2}| < \cdots < \beta^{n-N+1}|a_N|$

$n-(n-N)$

Thus $\quad |a_n| < \beta^{n-N}|a_N|$

Let $C = \beta^{-N}|a_N|$, a const. independent of n.

Thus $\quad |a_n| < \beta^n C$.

Then $\sum |a_n|$ and $\sum a_n$ converges by comparison with $\sum C\beta^n$, a conv. geom series.

Root Test Let $\sum a_k$ be an infinite series.
Let $\alpha = \limsup_{k\to\infty} \sqrt[k]{|a_k|}$, called the <u>exponential growth rate</u>.
If $\alpha < 1$ the series converges, if $\alpha > 1$ div, $\alpha = 1$ incon.

Pf If $\alpha < 1$, pick β s.t. $\alpha < \beta < 1$. Then $\exists K$, s.t. $k \geq K \Rightarrow |a_k|^{1/k} \leq \beta$. Thus $|a_k| \leq \beta^k$.

Then $\sum |a_k|$ converges by comparison to the geom series $\sum \beta^k$. Hence so does $\sum a_k$.

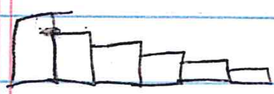
If $\alpha > 1$ the situation is similar.

The text gives examples showing $\alpha = 1$ can go either way. ∎

It is helpful when teaching calc II to pt out RR tests are generalizations of the geom series test, and that the p-series is most often used when these fail. ($\alpha = 1$).

Alt. Series If $a_k \geq 0$ and a_k decreasing with limit $a_k = 0$, then $\sum_{k=0}^{\infty} (-1)^k a_k$ converges.

Outline of proof.



Let $S_n = \sum_{k=0}^{n} (-1)^k a_k$.

Then (S_{2n}) is decreasing and bdd below (by $a_0 - a_1$)

Also (S_{2n+1}) is increasing bdd above by a_0

Let $L_1 = \lim S_{2n}$

$L_2 = \lim S_{2n+1}$

$S_{2n+1} - S_{2n} = a_{2n+1} \to 0$ so $L_1 = L_2$.

It also follows $|L - S_n| \leq a_{n+1}$ $\blacksquare$

Rearrangement Thm Let $\sum a_n$ be abs. conv. ~~Th~~ Let $\beta : \mathbb{N} \to \mathbb{N}$ be one-to-one and onto. Then $\sum a_{\beta(n)}$ conv. ~~abs~~ to the same ~~way~~ value.

Pf Let $y_n = x_{\beta(n)}$. Let $s_p = \sum_{n=1}^{p} x_n$, $t_q = \sum_{n=1}^{q} y_n$.

Let K be an upper bd for $\left(\sum_{n=1}^{p} |x_n|\right)_{p=1}^{\infty}$.

Take q as given for now.

Let $p = \max \beta^{-1}(n)$, $n = 1 \dots q$. Then

$$\sum_{n=1}^{q} |y_n| \le \sum_{n=1}^{p} |x_n| \le K, \ \forall q.$$

Thus $\sum |y_n|$ converges and hence $\sum y_n$ does too.

Let $x = \sum_{n=1}^{\infty} x_n$, $y = \sum_{n=1}^{\infty} y_n$. We claim $x = y$.

Let $\varepsilon > 0$. Let N be s.t. if $q > p \ge N$

then $|x - s_p| < \varepsilon$ and $\sum_{k=p+1}^{q} |x_k| < \varepsilon$ (cauchy crit.)

Fix p. Choose r st. $|y - t_r| < \varepsilon$ and st.

each $x_1, x_2 \dots x_p$ occurs in the terms for t_r.

Now choose $q > p$ st. every y_k in t_r is

also in s_q. Thus,

$$|x - y| \le |x - s_q| + |s_q - t_r| + |t_r - y| < \varepsilon + \sum_{p+1}^{q} |x_k| + \varepsilon < 3\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $x = y$. ∎

However, if $\sum a_k$ converges cond. (that is $\sum |a_k|$ ~~dive~~ diverges) then under rearrangements anything can happen and does!

$\exists\ \beta : \mathbb{N} \to \mathbb{N}$ (one-to-one, onto) st $\sum a_{\beta(n)}$ div. to $+\infty$, $-\infty$, or conv to any real number.

So exercises 64–67.

Series of Functions.

$$\sum f_k(x) \qquad \text{power series: } \sum c_k x^k.$$

Thm $\quad R = \dfrac{1}{\limsup\limits_{k\to\infty} |c_k|^{1/k}}$

$|x| < R$ conv.
$|x| > R$ div.
$|x| = R$??

pf: Use Root Test.