Ch 4 Function Spaces

Section 1

pt-wise convergence $f_n(x) \to f(x)$

uniform convergence $f_n \rightrightarrows f$.

Thm $f_n \rightrightarrows f$ and each f_n is cont. at x_0. Then f is cont. at x_0.

$f_n : [a,b] \to \mathbb{R}$.

pf Let $\varepsilon > 0$. $x_0 \in [a,b]$. $\exists N$ s.t. $n \geq N$

$\Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{3} \quad \forall x \in [a,b]$.

$\exists \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |f_N(x) - f_N(x_0)| < \frac{\varepsilon}{3}$.

Then

$$|f(x) - f(x_0)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| < \varepsilon.$$

∎

Def Let $C_b = C_b([a,b], \mathbb{R})$ = all bdd functions $[a,b] \to \mathbb{R}$.

sup norm $\|f\| = \sup\{|f(x)| : x \in [a,b]\}$.

It is a norm: $\|f\| \geq 0$, $\|f\| = 0 \Leftrightarrow f \equiv 0$.

$\|cf\| = |c|\,\|f\|$.

$\|f+g\| \leq \|f\| + \|g\|$.

This gives a metric $d(f,g) = \sup|f(x) - g(x)|$.

Thm Conv. wrt sup-metric is eq. to unif. conv.

pf: easy.

Thm C_b is a complete metric sp.

Pf: Let (f_n) be a Cauchy seq in C_b.
First we show it has a pt-wise limit.
Let $x_0 \in [a,b]$. Let $\varepsilon > 0$. Let N be s.t.
$n, m \geq N \Rightarrow d(f_n, f_m) < \varepsilon$. Then

$$|f_n(x_0) - f_m(x_0)| \leq \sup_{x \in [a,b]} \{ |f_n(x) - f_m(x)| \} = d(f_n, f_m) < \varepsilon.$$

Thus, $\lim_{n \to \infty} f_n(x_0)$ exists, for $\forall x_0 \in [a,b]$.

Define $f(x) = \lim_{n \to \infty} f_n(x)$.

Now we show the convergence is in fact uniform.
Let $\varepsilon > 0$. Let N_1 be s.t. $n, m \geq N_1 \Rightarrow$
Let $x \in [a,b]$.

$$d(f_n, f_m) < \frac{\varepsilon}{2}.$$

Let N_2 be s.t. $n \geq N_2 \Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{2}$.
~~Let $N = \max(N, N_2)$, and suppose~~ Let $n \geq N_1$, $m \geq \max(N_2, N_1)$

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon.$$

Thus the convergence in uniform.

Lastly, we need to show f is bdd.

$\exists N$ s.t. $|f_N(x) - f(x)| < 1, \ \forall x \in [a,b]$.

Since $f_N(x)$ is bdd, say by $|f_N(x)| \leq M$,

then $f(x)$ is bdd by $|f(x)| \leq M+1$.

Thus $f_n \rightrightarrows f \in C_b$. ■

Cor It follows that $C^0 \subset C^0_b$ is closed & complete

Pf: easy see textbook

Next we look a series of functions: $\sum_{k=0}^{\infty} f_k(x)$.

Thm Weierstrass M-test For $k = 0, 1, 2, \ldots$, let $f_k(x) \in C_b$ and $\| f_k \| \leq M_k$. If $\sum M_k$ converges, then $\sum f_k$ converges uniformly. (Pt-wise conv is obvious.)

Pf Let $F_n(x) = \sum_{k=0}^{n} f_k(x) \quad \forall x \in [a,b]$.

Let $n > m$. Then $d(F_n, F_m) \leq d(F_n, F_{n-1}) + \cdots + d(F_{m+1}, F_m)$

$$= \| F_n - F_{n-1} \| + \cdots + \| F_{m+1} - F_m \| = \| f_n \| + \cdots + \| f_{m+1} \| \leq \sum_{k=m+1}^{n} M_k$$

$\forall \epsilon > 0$, $\exists N$ s.t. $n > m \geq N \Rightarrow \sum_{k=m+1}^{n} M_k < \epsilon$. ■

Now we present some basic results related to integrability.

Thm: Let $f_n : [a,b] \to \mathbb{R}$ be R.I. ($f \in C_n$).
Suppose $f_n \rightrightarrows f$. Then

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f(x)\,dx$$

Here are two corollaries, then we'll do the proof.

- If $f_n \in \mathcal{R}$ and $f_n \rightrightarrows f$, then $\int_a^x f_n(t)\,dt \rightrightarrows \int_a^x f(x)\,dt$ $\forall x \in [a,b]$.
- If $\sum_{k=0}^{n} f_k \rightrightarrows F$, then $\sum \int_a^b f_n(x)\,dx = \int_a^b \sum f_n(x)\,dx$.

Proof of Thm: By the RL Thm ~~each~~ each $f_n \in C_b$ and if Z_n is the set of pts where f_n is not cont, it is a zero set.

By Thm 1, $\forall x \in [a,b] \setminus (\cup Z_n)$, f is cont.

Since $\cup Z_n$ is ~~countable~~ a zero set, ~~$f \in \mathcal{R}$~~. Since each f_n is bdd and C_b is complete (Thm 3) f is bdd, $\in C_b$.

Thus $f \in \mathcal{R}$.

Lastly, $\left| \int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx \right| = \left| \int_a^b f(x) - f_n(x)\,dx \right|$

$\|f_n - f\|$

$\le \int_a^b |f(x) - f_n(x)|\,dx \le (b-a)\, d(f, f_n) \to 0.$

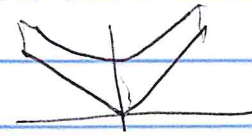
Thus $\int_a^b f_n \to \int_a^b f$. $\square$

The situation is different for diff'able functions.

Ex Let $f_n(x) = \sqrt{x^2 - \frac{1}{n}} : [-1,1] \to \mathbb{R}$.

$\lim f_n(x) = \sqrt{x^2} = |x|$, is not diff at $x=0$.

You can show the conv. is unif.



But we have the following

Thm 9 Suppose $f_n \rightrightarrows f$ and $f_n' \rightrightarrows g$. Then $f' = g$.

If each f_n' is cont., the proof is easy. see text.
Suppose this is not necessarily the case.

Pf Let

$$\phi_n(t) = \begin{cases} \dfrac{f_n(t) - f_n(x)}{t - x} & t \neq x \\ f_n'(x) & t = x \end{cases}$$

$$\phi(t) = \begin{cases} \dfrac{f(t) - f(x)}{t - x} & t \neq x \\ g(x) & t = x. \end{cases}$$

Each $\phi_n(t)$ is cont. (check at $t = x$ limit).

Clearly, $\phi_n(x) \to \phi(x)$. We will show $\phi_n \rightrightarrows \phi$.

$\forall m,n$ the MVT gives

$$\phi_m(t) - \phi_n(t) = \frac{[f_m(t) - f_n(t)] - [f_m(x) - f_n(x)]}{t - x} = f_m'(\theta) - f_n'(\theta)$$

for some θ between t and x. Since $f_n' \rightrightarrows g$,

$f_m' - f_n' \rightrightarrows g - g = 0$ as $m, n \to \infty$. You can use

this to show (ϕ_n) is Cauchy in C^0.

Since C^0 is complete $\exists \psi \in C^0$ s.t. $\phi_n \rightrightarrows \psi$.

Since $\phi_n(x) \to \phi(x)$, pt-wise $\psi(x) = \phi(x)$.

Since, $\psi(x)$ is cont., $\lim_{t \to x} \psi(t) = \psi(x)$ thus

$f'(x) = g(x)$. $\blacksquare$

Cor $\left(\sum f_k'\right) = \left(\sum f_k\right)'$ under corresponding assumptions.

Section 2 Power Series $\sum_{k=0}^{\infty} c_k x^k$.

Recall $R = \dfrac{1}{\limsup_{k\to\infty} |c_k|^{\frac{1}{k}}}$ is radius of convergence. typo: book leaves out $|\ |$..

Thm If $r < R$ then $\sum_{k=0}^{n} c_k x^k \rightrightarrows \sum_{k=0}^{\infty} c_k x^k$ on $[-r, r]$.

pf Choose $\beta \in (r, R)$. $\exists N$ s.t. $k \geq N \Rightarrow |c_k|^{\frac{1}{k}} < \frac{1}{\beta}$.

If $|x| \leq r$ then $|c_k x^k| \leq \left(\frac{r}{\beta}\right)^k$.

$\sum \left(\frac{r}{\beta}\right)^k$ is a conv. geom series. Hence by the M-test $\sum c_k x^k$ conv. unif on $[-r, r]$.

Thm A power series can be integrated and differentiated term-by-term on $(-R, R)$.

pf : see text.

Recall that a function is <u>analytic</u> if it is the limit of a power series.

<u>Thm 13</u> $C^\omega \subset C^\infty$.

<u>Idea of Pf</u> Differentiate the power series term-by-term.

<u>Section 3</u> : Compactness & Equicontinuity in C^0

In C^0 closed & bdd $\not\Rightarrow$ compact!
Even the closed unit ball is not compact

<u>Ex</u> $B = \{ f \in C^0([0,1] \to \mathbb{R}) \mid \|f\| \le 1 \}$.

Then $(x^n) \subset B$, but has no convergent subseq ($\underline{\quad}$ $\notin C^0$).

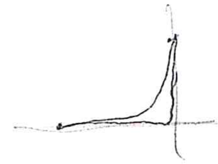
<u>Ex</u> Let $\mathbb{R}^\infty = \{ (x_i)_{i=1}^\infty \mid x_i \in \mathbb{R} \}$.

$\{ (1, 0, \dots 0 \dots), (0, 1, 0, \dots), (0\ 0, 1, 0, \dots), \dots \}$

is a seq in unit ball with no limit wrt sup norm.

(But with a different norm $\|(x_i)\| = \sum_{i=1}^\infty \frac{|x_i|}{2^i}$,
it ~~does~~ conv. to $(0, 0, \dots)$)

~~<u>Ex</u>: Let $\mathcal{F} = \{ f_r \mid r \in [0,1] \}$ be~~ ~~$f_r : [0,1] \to \mathbb{R}$~~ ~~$r-\delta$ r $r+\delta$~~ ~~$f = \frac{1}{10} \min(r, 1-r)$.~~



Def Let $\mathcal{E}$ be a subset of C^0. Then $\mathcal{E}$ is <u>equicontinuous</u> if $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$|s-t| < \delta \Rightarrow |f(s) - f(t)| < \varepsilon \quad \forall f \in \mathcal{E}.$$

Q: Show $\{x^n\}$ is not equicontr.

Big Thm <u>Arzela-Ascoli Thm</u>

Any bounded, equicont. seq (f_n) in $C^0([a,b]\to\mathbb{R})$ has a uniformly convergent ~~sub~~ seq.

Pf $\exists M$ s.t. $\|f_n\| < M \ \forall n = 1, 2, \dots$

Let $D = \{d_1, d_2, \dots\}$ be a countable dence subset of $[a,b]$.

For any i, $(f_n(d_i))$ is a bdd seq.

$\exists$ a subseq of $(f_n(d_1))$ that converges.

Say $f_{n_{1,k}}(d_1) \to y_1$ as $k \to \infty$.

Consider $(f_{n_{1,k}}(d_2))$. It has a conv. subseq

Say $f_{n_{2,k}}(d_2) \to y_2$ as $k \to \infty$.

Also $f_{n_{2,k}}(d_1) \to y_1$.

We can construct a family of subsequences $f_{m,k}$ s.t.

$(f_{m,k})$ is a subseq of $(f_{m-1,k})$

For $j \leq m$, $f_{m,k}(d_j) \to y_j$ as $k \to \infty$.

$\forall m \quad \exists K_m$ s.t. $k \geq K_m \Rightarrow$

$$|f_{m,k}(d_j) - y_j| < \frac{1}{m}$$

provided $j \leq m$. (We may require $K_{m+1} \geq K_m$.)

$f_{1,1}(d_1)$, $f_{1,2}(d_1)$, $f_{13}(d_1)$, ..., $f_{1K_1}(d_1)$, ... $\to y_1$

$f_{2,1}(d_2)$, $f_{22}(d_2)$, ... $f_{2K_2}(d_2)$, --- $\to y_2$

$f_{31}(d_3)$ $f_{32}(d_3)$, ... $f_{3K_3}(d_3)$, ... $\to y_3$

$\vdots \qquad \vdots$

$f_{m1}(d_m)$

Now take the "diagonal" subseq

$$g_m(x) = f_{m,K_m}(x), \quad m = 1, 2, \ldots$$

for each $x \in [a,b]$.

If $x = d_i$, for any i, then $g_m(d_i) \longrightarrow y_i$.

Why? Eventually $m \geq i$. Since $g_{m \geq i}(d_i)$ is a subseq of $f_{i,k}(d_i)$ it has the same limit, y_i.

Now we show that $g_m(x)$ converges at the other pts $x \in [a,b]$, and that the conv. is unif. We will do both by showing that (g_m) is a Cauchy seq in C^0.

Let $\varepsilon > 0$. By assumption, (f_n) is equicontinuous. $\exists\, \delta > 0$ s.t. $\forall\, s,t \in [a,b]$,

$$|s-t| < \delta \Rightarrow |g_m(s) - g_m(t)| < \frac{\varepsilon}{3}.$$

Cover $[a,b]$ with $\{(d_i - \delta, d_i + \delta),\ i = 1 \ldots\}$
Pick a finite subcover $\{(d_{i_k} - \delta, d_{i_k} + \delta),\ k = 1 \ldots n\}$.
Let $J = \max(i_k)$. We will work with $\{d_1, \ldots d_J\}$.

$\exists N$ s.t. for $p, q \geq N$ and $j \leq J$, we have

$$|g_p(d_j) - g_q(d_j)| < \frac{\varepsilon}{J}.$$

Given $p, q \geq N$ and $x \in [a,b]$, $\exists\, d_j$, $j \leq J$, s.t.

$|d_j - x| < \delta$. Then

$$\begin{aligned} |g_p(x) - g_q(x)| &\leq |g_p(x) - g_p(d_j)| + |g_p(d_j) - g_q(d_j)| \\ &\quad + |g_q(d_j) - g_q(d_j)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

"Hence (g_n) is Cauchy in C^0, it converges in C^0, and the proof is complete." $\square$.

<u>Ex</u> We can define $2^x = e^{x \ln 2}$, since we defined e^x as the inverse of $\ln x = \int_1^x \frac{1}{t}\,dt$.

But in pre-calc this is not realistic.

But, we can, at this level, define $2^{p/q}$.

Then we wave our hands and say $2^x = \lim_{\frac{p}{q} \to x} 2^{p/q}$,

But we are really using Arzela-Ascoli.

Cor 16 Let $f_n : [a,b] \to \mathbb{R}$, $n = 1,2,\dots$, be differentiable (hence cont. and bdd (individually)), and suppose $\exists M > 0$ s.t. $|f_n'(x)| \le M \ \forall x \in [a,b], n \in \mathbb{N}$. If $\exists x_0 \in [a,b]$ s.t. $|f_n(x_0)|$ is bdd for all $n \in \mathbb{N}$, then (f_n) has a subseq. that converges uniformly on $[a,b]$.

non-Ex 1 $\{x+n : [0,1] \to \mathbb{R} \mid n = 1,2,\dots\}$ satisfies all the conditions but the last and clearly there is no conv. subseq.

non-Ex 2 $\{nx \dots\}$ satisfies the last condition $(x_0 = 0)$ but the ~~df~~ derivatives are unbdd. Again, no conv. subseq. exists.

Pf Let $\varepsilon > 0$ be given and let $\delta = \frac{\varepsilon}{M+1}$. For $s < t$, both in $[a,b]$, the MVT says $\exists \theta \in (s,t)$ s.t.

$$\frac{f_n(t) - f_n(s)}{t - s} = f_n'(\theta).$$

Thus, if $|t-s| < \delta$, then $|f_n(t) - f_n(s)| < (t-s)|f_n'(\theta)| < \delta M < \varepsilon$. Since δ did not depend on s, t or n, (f_n) is equicont. (Note: θ depended on n, but this does not matter.)

Let C be a bd for $|f_n(x_0)|$. Then

$$|f_n(x)| \le |f_n(x) - f_n(x_0)| + |f_n(x_0)| \le |x - x_0| M + C \le (b-a)M + C.$$

Thus (f_n) is uniformly bdd. By AA (f_n) has a unif. conv. subseq. ∎

Thm 17 Heine-Borel Theorem in a function space.
A subset $\mathcal{E} \subset C^0$ is compact iff it is closed, bdd, and equicontinuous.

The proof uses the idea of "total boundedness" from Ch 2, pg 92, that we didn't cover.

Def A subset A of a metric space M is totally bdd if $\forall \varepsilon > 0$ $\exists$ finite subset of A, $\{a_1, \dots a_n\}$ s.t. $\{B(a_i, \varepsilon) \mid i = 1 \dots n\}$ covers A.

Thm 56 from Ch 2: (Generalized H-B Thm)
A subset of a complete metric sp is compact iff it is closed and totally bdd.

Pf: See text, pg 92-93.

Pf: of HBT in fnc. sp.
The A.A.T gives one direction. Suppose $\mathcal{E}$ is closed, bdd and equi. If (f_n) is a seq. in $\mathcal{E}$, some subseq (f_{n_k}) conv. unif to a limit, by AAT. The limit lies in $\mathcal{E}$ since $\mathcal{E}$ is closed.
Thus $\mathcal{E}$ is seq. compact.

For the other direction, assume $\mathcal{E}$ is compact. By Thm 2.56 it is closed and totally bdd.

Let $\varepsilon > 0$, and pick any $f \in \mathcal{E}$.

$\exists\, f_1, \ldots f_n \in \mathcal{E}$ s.t. $B(f_k, \frac{\varepsilon}{3})$, $k = 1, \ldots n$ cover $\mathcal{E}$.

Since each f_k is unif. cont. $\exists\, \delta > 0$ s.t.

$$|s - t| < \delta \Rightarrow |f_k(s) - f_k(t)| < \frac{\varepsilon}{3}, \quad k = 1, \ldots n.$$

For some k, $f \in B(f_k, \frac{\varepsilon}{3})$. Thus

$$|s - t| < \delta \Rightarrow |f(s) - f(t)| \leq |f(s) - f_k(s)| + |f_k(s) - f_k(t)| + |f_k(t) - f(t)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus, $\mathcal{E}$ is eq. cont. $\blacksquare$

Section 4 → Uniform Approximation in C^0

Thm Weierstrass Approx. Thm.
The set of polynomials is dense in $C^0([a,b] \to \mathbb{R})$

The textbook gives two proofs. I'll cover the first in class and you should read/study the second as well.

The first proof uses Bernstein polynomials. They are also used in computer graphics to create Bézier curves. We will restrict our attention to the interval $[0,1]$ for simplicity.

For each $n \in \mathbb{N} \cup \{0\}$ there are $n+1$ Bernstein basis poly's of degree n. They are

$$b_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, 2, \ldots, n,$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. For example

$b_{00}(x) = 1$ (so n can be zero.)
$b_{01}(x) = (1-x)$ $\quad b_{11}(x) = x$
$b_{02}(x) = (1-x)^2$ $\quad b_{12} = 2x(1-x)$, $b_{22} = x^2$
etc.

A poly $B_n(x)$ of the form $= \sum_{k=0}^{n} c_k b_{kn}(x)$ is called a Bernstein poly of degree n.

A useful fact is that for each n,

$$\sum_{k=0}^{n} b_{k,n}(x) = 1. \qquad (*)$$

We say they form a "partition of unity". This follows from the binomial thm:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \qquad (bt)$$

Let $y = 1-x$.

Another identity we will use is

$$\sum_{k=0}^{n} (k-nx)^2 b_{k,n}(x) = nx(1-x). \qquad (\#)$$

The proof of this is a little harder. It goes like this. Take $\frac{\partial}{\partial x}$ of both sides of (bt), twice,

$$n(x+y)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1} y^{n-k}$$

$$n(n-1)(x+y)^{n-2} = \sum_{k=0}^{n} \binom{n}{k} k(k-1) x^{k-2} y^{n-k}.$$

Now let $y = 1-x$ in both of these.

$$n = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1} (1-x)^{n-k}$$

$$n(n-1) = \sum_{k=0}^{n} \binom{n}{k} k(k-1) x^{k-2} (1-x)^{n-k}$$

Multiply the first by x and the second by x^2.

$$nx = \sum_{k=0}^{n} \binom{n}{k} k\, x^k (1-x)^{n-k} = \sum_{k=0}^{n} k\, b_{k,n}(x)$$

$$n(n-1)x^2 = \sum_{k=0}^{n} \binom{n}{k} k(k-1) x^k (1-x)^{n-k} = \sum_{k=0}^{n} k(k-1)\, b_{kn}(x).$$

$$= \sum_{k=0}^{n} k^2 b_{kn}(x) - \underbrace{\sum_{k=0}^{n} k\, b_{kn}(x)}_{(=nx)}.$$

Hence

$$\sum_{k=0}^{n} k^2 b_{k,n}(x) = n(n-1)x^2 + \sum k\, b_{kn}(x)$$
$$= n(n-1)x^2 + nx.$$

Now, go back to the claim (#)

$$\sum_{k=0}^{n} (k-nx)^2 b_{kn}(x) = \sum_{k=0}^{n} k^2 b_{kn}(x) - 2nx \sum_{k=0}^{n} k\, b_{kn}(x) + n x^2 \sum_{k=0}^{n} b_{kn}(x)$$

$$= n(n-1)x^2 + nx - 2nx(nx) + nx^2$$

$$= \cdots = nx(1-x). \checkmark$$

Proof of ~~#~~18 (W. A. Thm) (on [0,1])

For each $n \in \mathbb{N}$, Let $p_n(x) = \sum_{k=0}^{n} \binom{n}{k} c_k x^k (1-x)^{n-k}$,

$= \sum_{k=0} c_k b_{k,n}(x)$.

where $c_k = f\left(\frac{k}{n}\right)$.

Note, $f(x) = f(x) \cdot 1 = f(x) \sum b_{kn}(x) = \sum_{k=0}^{n} f(x) b_{kn}(x)$.

We will show $p_n \rightrightarrows f$. Notice that as $n \to \infty$

the partition $\{\frac{0}{n}, \frac{1}{n}, \dots \frac{n}{n}\}$ gets finer. Fix a value of $x \in [0,1]$ for now. Let $\varepsilon > 0$.

Since $f(x)$ is unif. cont. on $[a,b]$, $\exists \delta > 0$ s.t.

$$|t - s| < \delta \Rightarrow |f(t) - f(s)| < \varepsilon/2.$$

Define

$$K_{1,n} = \{k \in \{0,1,\dots,n\} \mid |\tfrac{k}{n} - x| < \delta\}, \quad K_{2,n} = \{0, \dots n\} \setminus K_{1,n}.$$

Now

$$|p_n(x) - f(x)| = \left|\sum (c_k - f(x))\, b_{k,n}(x)\right| \le \sum_{k=0}^{n} |c_k - f(x)|\, b_{kn}(x)$$

$$= \sum_{k \in K_{1n}} |c_k - f(x)|\, b_{kn}(x) + \sum_{k \in K_{2,n}} |c_k - f(x)|\, b_{kn}(x)$$

Since $c_k = f(\frac{k}{n})$, $|c_k - f(x)| \leq \varepsilon/2$ for $x \in K_{1,n}$.

Thus $\sum_{k \in K_{1,n}} |c_k - f(x)| b_{kn}(x) \leq \frac{\varepsilon}{2} \sum_{K_1} b_{k,n}(x) \leq \frac{\varepsilon}{2} \sum b_{kn}(x) = \frac{\varepsilon}{2}$

For $k \in K_{2,n}$ we have $|\frac{k}{n} - x| \geq \delta \Rightarrow |k - nx|^2 \geq (n\delta)^2$.

Now by (#),

$$n\,x(1-x) = \sum_{k=0}^{n} (k-nx)^2 b_{kn}(x) \geq \sum_{k \in K_{2n}} (k-nx)^2 b_{kn}(x)$$

$$\geq \sum_{k \in K_2} (n\delta)^2 b_{kn}(x).$$

Therefore,
$$\sum_{k \in K_2} b_{kn}(x) \leq \frac{n\,x(1-x)}{(n\delta)^2} \leq \frac{1}{4n\delta^2}$$

since max of $x(1-x)$ is $\frac{1}{4}$ over $[0,1]$.

Let $M = \|f\|$. Then the factors $\|\underbrace{f(\tfrac{k}{n})}_{c_k} - f(x)\| \leq 2M$.

Thus,
$$\sum_{K_2} |c_k - f(x)| b_{kn}(x) \leq \frac{M}{2n\delta^2} \leq \frac{\varepsilon}{2}$$

for large enough n. Thus $\exists N$. s.t. $n \geq N$

$\Rightarrow |p_n(x) - f(x)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, and N did not depend on x? But why is it indep of x??

The book does not say. So let's think about it. k_1 & k_2 depend on x.

But the bounds we get on $\sum_{k_1}$ & $\sum_{k_2}$

did not depend on k_1 or k_2. $\square$

Bernstein Polynomial Approximations

Find the first value of n so the the n^{th} Bernstein polynomial for f(x) = |x−0.5| has $||p_n - f|| < 0.1$.

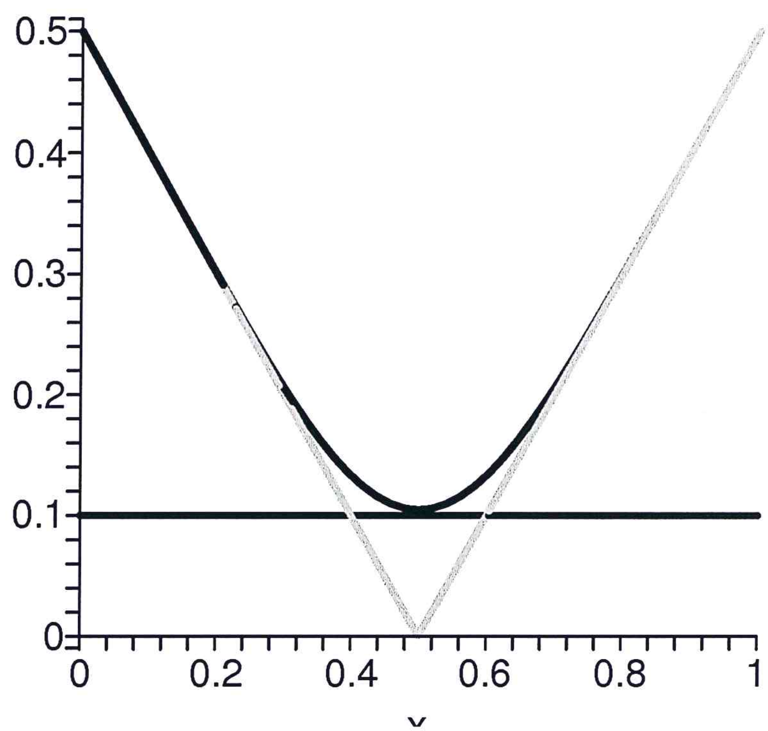
```
> pn:= x-> sum((n!/k!/(n-k)!)*abs(k/n-0.5)*x^k*(1-x)^(n-k),k=0..n);
```

$$pn := x \to \sum_{k=0}^{n} \frac{n! \left| \frac{k}{n} - 0.5 \right| x^k (1-x)^{(n-k)}}{k!\,(n-k)!}$$

```
> n:=15;
```

$$n := 15$$

```
> plot([0.1,abs(x-0.5),pn(x)],x=0..1,color=[black,gray,black],
  linestyle=[2,1,1],thickness=2);
```




```
> pn(0.5);
```

$$0.1047363281$$

```
> n:=16;
```

$$n := 16$$

```
> pn(0.5);
```

$$0.09819030762$$

The answer is n=16.

Repeat for f(x) = | |x–0.5| – 0.25| and $||p_n - f||<0.02$.

```
> f:=x-> abs(abs(x-0.5)-0.25);
```

$$f := x \to |\,|x - 0.5| - 0.25|$$

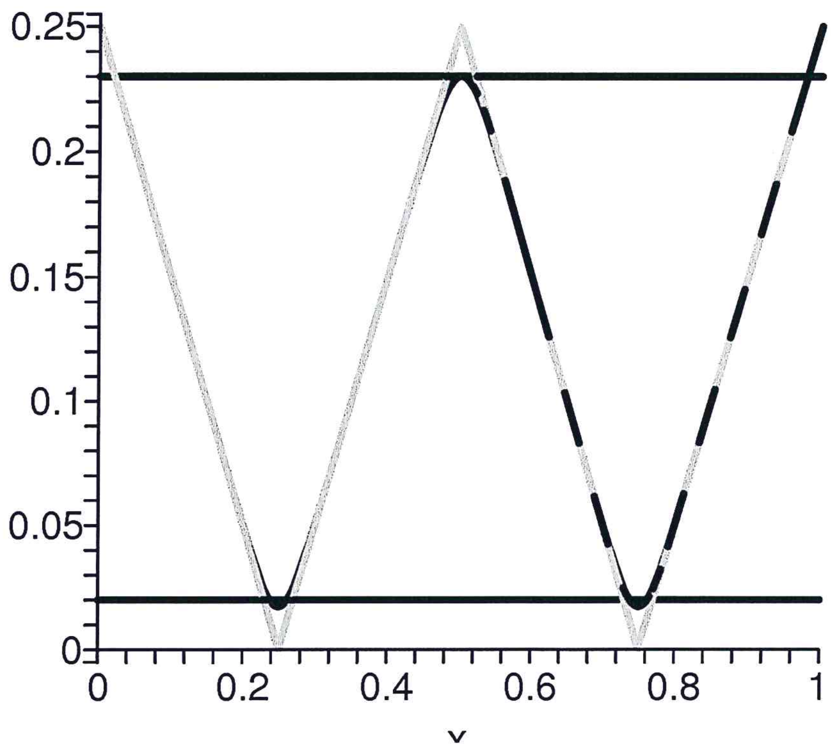
```
> pn:= x-> sum((n!/k!/(n-k)!)*f(k/n)*x^k*(1-x)^(n-k),k=0..n);
```

$$pn := x \to \sum_{k=0}^{n} \frac{n!\, f\left(\frac{k}{n}\right) x^k (1-x)^{(n-k)}}{k!\,(n-k)!}$$

```
> n:=398;
```

$$n := 398$$

```
> plot([0.02,0.23,f(x),pn(x)],x=0..1,color=[black,black,gray,black],
  linestyle=[2,2,1,1],thickness=2);
```




```
> pn(0.5);
```

$$0.2300153875$$

The answer is n=398.

Stone-Weierstrass Thm

Let M be a complete compact metric space.
Let $\mathcal{A} \subset C^0(M, \mathbb{R})$. Then $\mathcal{A}$ is called a
function algebra if $\forall f, g \in \mathcal{A}$, $c \in \mathbb{R}$ we have

$$f+g,\ cf,\ f \cdot g \text{ are in } \mathcal{A}.$$

A function algebra vanishes at $p \in M$ if $\forall f \in \mathcal{A}$, $f(p) = 0$.

A function alg separates points if $\forall p_1, p_2 \in M$, $p_1 \neq p_2$, $\exists f \in \mathcal{A}$ s.t. $f(p_1) \neq f(p_2)$.

Ex Polynomials in $C^0(\mathbb{R}, \mathbb{R})$ is a nonvanishing func alg that sep. pts.

Ex $\{ p(x)(x-3) \mid p(x) \text{ any poly} \}$ vanishes at $p = 3$.
$\{ p(x)(1-x^2) \}$ does not sep. 1 and -1.

~~Ex $\{ c \sin^n(x) \}$ does not sep. $0, \pi, 2\pi, \ldots$~~

Thm SW Thm. Let M be compact metric space, $\mathcal{A} \in C^0(M, \mathbb{R})$ a func. alg. that vanishes nowhere and sep. pt., then $\mathcal{A}$ is dense in $C^0(M, \mathbb{R})$.

First some lemmas.

Lemma If $\mathcal{A}$ vanishes nowhere and sep. pt, then $\forall p_1, p_2 \in M$, $p_1 \neq p_2$ and $c_1, c_2 \in \mathbb{R}$, $\exists f \in \mathcal{A}$ s.t.

$$f(p_1) = c_1 \text{ and } f(p_2) = c_2.$$

Pf $\exists g_1, g_2 \in \mathcal{A}$ s.t. $g_1(p_1) \neq 0$ and $g_2(p_2) \neq 0$.
Let $g(x) = (g_1(x))^2 + (g_2(x))^2$. Then $g \in \mathcal{A}$, and $g(p_1) \neq 0$, $g(p_2) \neq 0$.

~~Al~~ $\exists h \in \mathcal{A}$ s.t. $h(p_1) \neq h(p_2)$. Consider the matrix

$$H = \begin{bmatrix} a & ab \\ c & cd \end{bmatrix} = \begin{bmatrix} g(p_1) & g(p_1)h(p_1) \\ g(p_2) & g(p_2)h(p_2) \end{bmatrix}$$

Now $a \neq 0$, $c \neq 0$, and $b \neq d$. ~~Since~~ Thus

$$\det H = acd - abc = ac(b-d) \neq 0.$$

Therefore $\exists!$ solution to

$$\begin{bmatrix} a & ab \\ c & cd \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Let $f = p\,g(x) + q\,g(x)h(x)$. $f \in \mathcal{A}$.

$f(p_1) = pa + qab = c_1$ and

$f(p_2) = pc + qcd = c_2$. $\square$

Lemma ~~The closure~~ Let $\mathcal{A}$ be a func. alg. Then $\bar{\mathcal{A}}$ in $C^0(M, \mathbb{R})$ is a func. alg.

Pf Let $f, g \in \bar{\mathcal{A}}$, $c \in \mathbb{R}$.

$\exists\, f_n, g_n \in \mathcal{A}$ s.t. $f_n \to f$, $g_n \to g$.

Thus $(f_n + g_n) \to f+g \Rightarrow f+g \in \bar{\mathcal{A}}$.

$$(cf_n) \to cf \Rightarrow cf \in \bar{\mathcal{A}}$$

$$f_n \cdot g_n \to fg \Rightarrow f \cdot g \in \bar{\mathcal{A}}.$$

Thus $\bar{\mathcal{A}}$ is a func. alg. $\square$

Lemma $f \in \bar{\mathcal{A}} \Rightarrow |f| \in \bar{\mathcal{A}}$. $(f \in \mathcal{A} \not\Rightarrow |f| \in \mathcal{A})$.

Let $f \in \bar{\mathcal{A}}$. Let $M = \|f\|$.

Pf Let $\varepsilon > 0$. By the W.A.T., $\exists\, p$ s.t.

$$\sup\{\,|p(x) - |x|| : |x| \le M\,\} < \frac{\varepsilon}{2}$$

(We are applying WAT to $C^0([-M, M] \to \mathbb{R})$.)

The constant term of $p(x)$ is $|\ | \le \frac{\varepsilon}{2}$. Let

$q(x) = p(x) - p(0)$. Then $\sup\{\,|q(x) - |x|| : |x| \le M\,\} < \varepsilon$.

Let $q(y) = a_1 y + a_2 y^2 + \cdots + a_n y^n$. Define

$$g(x) = a_1 f(x) + a_2 (f(x))^2 + a_3 (f(x))^3 + \cdots + a_n (f(x))^n.$$

By ~~first~~ last lemma $g \in \bar{\mathcal{A}}$. Let $y = f(x)$. Then

$$\big| g(x) - |f(x)| \big| = \big| q(y) - |y| \big| < \varepsilon.$$

Since this ~~holds~~ can be done for any $\varepsilon > 0$ and $\bar{\mathcal{A}}$ is closed,

$|f| \in \bar{\mathcal{A}}$. $\square$

<u>Lemma</u> Let $f, g \in \bar{\mathcal{A}}$. Then $\max(f,g)$ and $\min(f,g)$ are in $\bar{\mathcal{A}}$.

<u>pf</u>

$$\max(f,g) = \frac{f+g}{2} + \frac{|f-g|}{2} \in \bar{\mathcal{A}}$$

$$\min(f,g) = \frac{f+g}{2} - \frac{|f-g|}{2} \in \bar{\mathcal{A}}. \quad \square$$

But induction, this extends to

max, min of $(f_1, f_2, \ldots f_n)$.

pf of SWT

Let $F \in C^0(M, \mathbb{R})$ and let $\varepsilon > 0$.
We claim $\exists\ G \in \bar{\mathcal{A}}$ s.t.

$$F(x) - \varepsilon < G(x) \leq F(x) + \varepsilon.$$

Since this will hold $\forall\ \varepsilon > 0$, $F \in \bar{\bar{\mathcal{A}}} = \bar{\mathcal{A}}$.
Hence $\bar{\mathcal{A}} = C^0(M, \mathbb{R})$.

Pick any two distinct pts $p, q \in M$. $\exists\ H \in \mathcal{A}$ s.t. $H(p) = F(p)$ and $H(q) = F(q)$.

At $x = q$ $\varepsilon + F(x) - H(x) > 0$. $\exists$ nbhd U_q of q s.t.

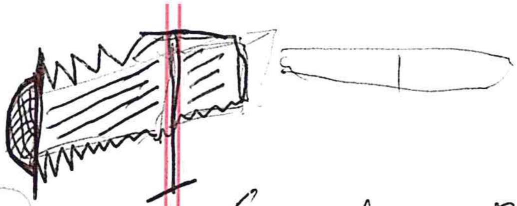
$x \in U_q \Rightarrow \varepsilon + F(x) - H(x) > 0$, or $F(x) - \varepsilon < H(x)$,

by the cont. of $\varepsilon + H(x) - F(x)$ as a function of x.

Let q vary over M, and for each select H_q and U_q. Finitely many U_q, say $U_{q_1}, \dots U_{q_n}$ cover M (compactness). Define

$$G_p(x) = \max\left(H_{pq_1}(x), \dots, H_{q_n}(x)\right).$$

$G_p \in \bar{\mathcal{A}}$, $G_p(p) = F(p)$ and $F(x) - \varepsilon < G_p(x)$ $\forall x \in M$.



Consider $-G_p(x) + F(x) + \epsilon$, as a cont. func of x.

At $x = p$ $\quad -G_p(p) + F(p) + \epsilon = \epsilon > 0$.

$\exists V_p$ s.t. $x \in V_p$ $\quad -G_p(x) + F(x) + \epsilon > 0$.

Then $\quad G_p(x) < F(x) + \epsilon \quad \forall x \in V_p$.

Let p vary over M. Then the V_p will cover M. $\exists$ a finite subcover, $V_{p_1}, V_{p_2}, \dots V_{p_n}$, and for each there is a corresponding G_{p_i} such $\forall x \in V_{p_i}$

$G_{p_i}(x) < F(x) + \epsilon$, but $F(x) - \epsilon < G_{p_i}(x)$ $\forall x \in M$.

Let $G(x) = \min(G_{p_1}(x), \dots G_{p_n}(x))$.

Then $G \in \bar{\mathcal{A}}$ and

$$F(x) - \epsilon < G(x) < F(x) + \epsilon$$

$\forall x \in M$. $\blacksquare$

Section 5: Contractions and Diff Eq.

<u>Def</u> Let $f: M \to M$. If $p \in M$ is s.t. $f(p) = p$, p is called a <u>fixed point</u> of f.

<u>Def</u> Let $f: M \to M$. If $\exists_{0 \le k < 1}$ s.t. $d(f(x), f(y)) \le k\, d(x,y)$ $\forall x, y \in M$, then f is called a <u>contraction</u>.

<u>Thm</u> [Banach Contraction Principle or the Contraction Mapping Thm] Let M be a complete metric sp. Let $f: M \to M$ be a contraction. Then f has a unique fixed pt.

<u>Pf</u> Uniqueness is easy, so we'll do this first. Suppose p, q are fixed pts. Then

$$d(f(p), f(q)) = d(p, q) < d(p, q).$$

Contradiction.

Choose any $x_0 \in M$. Let $x_n = f^n(x)$ $(f^n = f \circ f \circ \cdots \circ f)$. Then for any $n \in \mathbb{N}$

$$d(x_n, x_{n+1}) \le k\, d(x_{n-1}, x_n) \le k^2 d(x_{n-2}, x_{n-1}) \le \cdots \le k^n d(x_0, x_1).$$

In fact we can show (x_n) is Cauchy.

Let $m \leq n$

$$d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n)$$

$$\leq k^m d(x_0, x_1) + k^{m+1} d(x_0, x_1) + \cdots + k^{n-1} d(x_0, x_1)$$

$$= k^m (1 + k + \cdots + k^{n-m-1}) d(x_0, x_1)$$

$$\leq k^m \left(\sum_{l=0}^{\infty} k^l \right) d(x_0, x_1) = \frac{k^m}{1-k} d(x_0, x_1)$$

Let $\varepsilon > 0$. Since $k \in (0,1)$ $\exists N$ s.t. $m \geq N$

$$\Rightarrow \frac{k^m}{1-k} d(x_0, x_1) < \varepsilon.$$

Then (x_n) is Cauchy. Since M is complete $\exists p \in M$ s.t. $x_n \to p$. This is our fixed pt.

$$d(p, f(p)) \leq d(p, x_n) + d(x_n, f(x_n)) + d(f(x_n), f(p))$$

$$\leq d(p, x_n) + k^n d(x_0, x_1) + k\, d(x_n, p)$$

$$\longrightarrow 0 \quad \text{as } n \to \infty.$$

Thus $d(p, f(p)) = 0$ and $f(p) = p$. $\square$

Thm (Brouwer Fixed Point Thm) Suppose $f: B^m \to B^m$ is continuous where B^m is the closed unit ball in $\mathbb{R}^m$. Then f has a fixed point.

pf Text does $m=1$ case. For the rest, take Math 430.

ODE's: Picard's Existence Thm.

Let $F: \mathbb{R}^m \to \mathbb{R}^m$, be regarded as a vecto field. A <u>solution curve</u> is a curve $\gamma: \mathbb{R} \to \mathbb{R}^m$ such that

$$\gamma'(t) = F(\gamma(t)). \qquad (*)$$

If $\gamma(0)=p$, we say γ is a solution to F with <u>initial condition</u> $\gamma(0)=p$.

This can be generalized to manifolds, where now $F: M \to \mathbb{R}^m$ (M has $\dim = m$) and $\gamma: \mathbb{R} \to M$. But the idea of a derivative has to be generalized. This is done in courses on Differential Topology or Diff. Geom.

A v.f. F satisfies a <u>Lipschitz condition</u> if $\exists$ $L \geq 0$ s.t.

$$|F(x) - F(y)| \leq L|x-y|.$$

Lip $\Rightarrow$ cont., but not much more. Any diff vector field with a bdd derivative is Lip.

The eq. (*) + init. condition can be rewritten as

$$\gamma(t) = p + \int_0^t F(\gamma(s))\, ds$$

where the integration is defined to be the vector of the intergals of each component. See text for details.

~~**Thm** (Picard's Thm). Let F be a v.f. defined on $U \subset \mathbb{R}^n$, an open with $p \in U$. Then $\exists\ \gamma : (a,b) \to U$ s.t.~~

$$\gamma'(t) = F(\gamma(t)) \quad \text{and} \quad \gamma(0) = p$$

Thm (Picard's Thm) Let F be a Lip. v.f. defined on an open set $U \subset \mathbb{R}^m$, ~~an op~~ and let $p \in U$. Then $\exists\ a < 0 < b$ and $\gamma : (a,b) \to U$, that solve $\gamma'(t) = F(\gamma(t))$, $\gamma(0) = p$, and is unique on (a,b).

Pf Since F is Lip it is continuous. Thus on any ~~cpt~~ compact nbhd of p, N; there is a constant M s.t. $|F(x)| \leq M \ \forall x \in N$. Assume N is $\overline{B_r(p)}$. Let L be a Lip. constant for F over U. Choose $\tau > 0$ s.t.

$$\tau M \leq r \quad \text{and} \quad \tau L < 1.$$

Let $\mathcal{C} = \{ \text{cont. functions } \gamma : [-\tau, \tau] \to N \} = C^0([-\tau,\tau], N)$
Give $\mathcal{C}$ the sup metric:

$$d(\gamma_1, \gamma_2) = \sup\{ |\gamma_1(t) - \gamma_2(t)| : t \in [-\tau, \tau] \}.$$

Then $\mathcal{C}$ is a complete metric space. (Why?) (see pg 206.). Given $\gamma \in \mathcal{C}$, define

$$\underline{\Phi}(\gamma) = p + \int_0^t F(\gamma(s))\, ds.$$

Then, $\Phi : C^0([-\tau,\tau], \overset{N}{\cancel{\mathbb{R}}}) \to C^0([-\tau,\tau], \overset{N}{\cancel{\mathbb{R}}})$.
we will show

A fixed point (curve!) of Φ, $\Phi(\gamma) = \gamma$,

is a solution curve! We will show Φ

is a contraction.

First we check that $\Phi : \mathcal{C} \to \mathcal{C}$.

$$\left| \Phi(\gamma)(t) - p \right| = \left| \int_0^t F(\gamma(s))\, ds \right| \leq \tau M \leq r.$$

Thus $\Phi(\gamma)[-\tau,\tau] \subset N$, so $\Phi(\gamma) \in \mathcal{C}$.

It Φ is a contraction since

$$d(\Phi(\gamma_1), \Phi(\gamma_2)) = \sup_{t \in [-\tau,\tau]} \left| \int_0^t F(\gamma_1(s)) - F(\gamma_2(s))\, ds \right|$$

$$\leq \tau \sup_{s \in [-\tau,\tau]} \left| F(\gamma_1(s)) - F(\gamma_2(s)) \right|$$

$$\leq \tau \sup_s L \left| \gamma_1(s) - \gamma_2(t) \right| \leq \tau L\, d(\gamma_1, \gamma_2).$$

But $\tau L < 1$, so Φ contracts. Thus it has a unique fixed "point". ∎

Notice this proof gives us a method for finding the solution: We make a guess, and then apply Φ over and over.

Ex Let $F(x) = \frac{1}{2}x$, $x \in \mathbb{R}$, be a vector field on $\mathbb{R}$. Find $\gamma(t)$ s.t. $\gamma'(t) = F(\gamma(t)) = \frac{1}{2}\gamma(t)$, $\gamma(0) = 1$.

[You should know the answer is $e^{t/2}$.]

Sol F is Lip with $L = \frac{1}{2}$. For a guess, let $\gamma_0(t) = \phi$. Then

$$\cancel{\gamma_1(t) = 1 + \int_0^t \tfrac{1}{2}\cdot 0 \, ds = 1}, \quad \gamma_1(t) = 1 + \int_0^t \tfrac{1}{2}\cdot 1 \, ds = 1 + \frac{t}{2}$$

$$\gamma_2(t) = 1 + \frac{1}{2}\int_0^t 1 + \frac{s}{2} \, ds = 1 + \frac{t}{2} + \frac{t^2}{4\cdot 2}$$

$$\gamma_3(t) = 1 + \frac{1}{2}\int_0^t 1 + \frac{s}{2} + \frac{s^2}{4\cdot 2} \, ds = 1 + \frac{t}{2} + \frac{t^2}{4\cdot 2} + \frac{t^3}{8\cdot 2\cdot 3}$$

$$\gamma_4(t) = 1 + \frac{1}{2}\int_0^t 1 + \frac{s}{2} + \frac{s^2}{4\cdot 2} + \frac{s^3}{8\cdot 3\cdot 2} \, ds = 1 + \frac{t}{2} + \frac{t^2}{4\cdot 2} + \frac{t^3}{8\cdot 3!} + \frac{t^4}{16\cdot 4!}$$

$$\vdots$$

$$\gamma_n(t) = 1 + \left(\frac{t}{2}\right) + \frac{\left(\frac{t}{2}\right)^2}{2!} + \frac{\left(\frac{t}{2}\right)^3}{3!} + \frac{\left(\frac{t}{2}\right)^4}{4!} + \cdots + \frac{\left(\frac{t}{2}\right)^n}{n!}$$

Hence $\gamma_n(t) \longrightarrow e^{t/2}$.

Section 6: Analytic Functions

Let $f:(a,b)\to\mathbb{R}$. Recall f is <u>analytic</u> at $x_0\in(a,b)$ if there is a power series $\sum c_k(x-x_0)^k$ and $\delta>0$ s.t.

$$f(x)=\sum_{k=0}^{\infty} c_k(x-x_0)^k \quad \text{for} \quad |x-x_0|<\delta.$$

It follows that an analytic function at x_0 is C^∞ (smooth) at x_0. (Thm 13, Ch 4, Section 2, pg 212). We also know, that not every C^∞ function is analytic. (Exercise 14, Ch 3, pg 187-8).

We want to understand under what conditions will a smooth function be analytic.

Recall, for a power series, the radius of convergence is

$$R=\left(\limsup_{k\to\infty}\sqrt[k]{|c_k|}\right)^{-1} \qquad \text{(Thm 44, pg 185)}$$

Recall, that for a smooth (C^∞) func. the Taylor series centered at x_0 is

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

We know then that $R=\left(\limsup_{k\to\infty}\sqrt[k]{\frac{|f^{(k)}(x_0)|}{k!}}\right)^{-1}$, but

does the series converge to f? Not always as we have seen. When does it?

Let $\sigma > 0$ be s.t. $[x_0 - \sigma, x_0 + \sigma] \subset (a,b)$. Let $M_k = \max |f^{(k)}(x)|$ over $[x_0-\sigma, x_0+\sigma]$. Then the

<u>derivative growth rate</u> of f or is defined to be

$$\alpha = \limsup_{k \to \infty} \sqrt[k]{\frac{M_k}{k!}}.$$

Clearly, $\frac{1}{\alpha} \le R$.

Thm 25 Let $f : (a,b) \to \mathbb{R}$ be smooth. Let $x_0 \in (a,b)$ Let $\sigma > 0$ be s.t. $[x_0 - \sigma, x_0 + \sigma] \subset (a,b)$. If $\alpha\sigma < 1$, then the Taylor series of f convergences uniformly to f on $[x_0-\sigma, x_0+\sigma]$.

Pf Let $\delta > 0$ be s.t. $(\alpha + \delta)\sigma < 1$. Recall that Taylor's remainder formula says $\exists\, \theta \in (x_0, x)$ (or (x, x_0))

s.t. $$f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = \frac{f^{(n)}(\theta)}{n!}(x-x_0)^n$$

(Thm 11;5 pg 150). Then

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \right| \le \frac{M_n}{n!}\sigma^n = \left(\left(\frac{M_n}{n!}\right)^{\frac{1}{n}}\sigma\right)^n \le ((\alpha+\delta)\sigma)^n$$ ← big enough n.

$\forall \varepsilon > 0$, $\exists N$ s.t. $n \ge N \Rightarrow ((\alpha+\delta)\sigma)^n < \varepsilon$. Thus

$$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \rightrightarrows f(x), \quad \text{on } [x_0-\sigma, x_0+\sigma]. \quad \blacksquare$$

1692–1770

The next result uses two limits, that are related to ~~John~~ (James) Stirling's Formula. (See handout?) on webpage K. Conrad

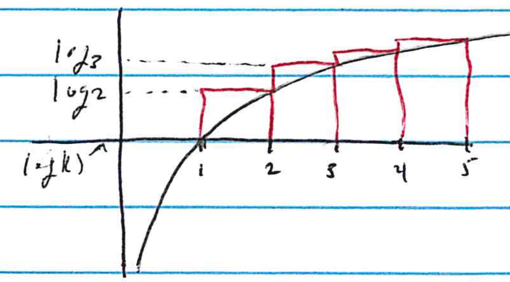
$$\boxed{n! \approx \sqrt{2\pi n}\, n^n e^{-n}}$$

1700's

I $\lim_{k\to\infty} \sqrt[k]{\frac{k^k}{k!}} = e.$

Pf: $\log\left(\sqrt[k]{\frac{k^k}{k!}}\right) = \frac{1}{k}\left(\log k^k - \log k!\right) = \log k - \frac{1}{k}\left(\log(k) + \log(k-1) + \cdots + \log(1)\right)$

Compare the graphs:



$$\sum_{n=1}^{k} \log(n) > \int_1^k \log(x)\,dx$$

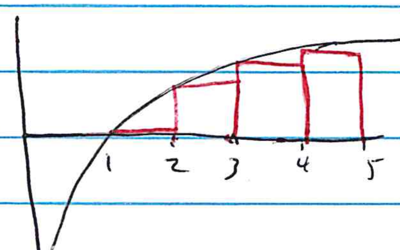
Thus $\log k - \frac{1}{k}\left(\log(k) + \log(k-1) + \cdots + \log(1)\right) < \log k - \frac{1}{k}\int_1^k \log(x)\,dx$

$= \log k - \frac{1}{k}\left(x\log x - x\right)\Big|_1^k = \log k - \frac{1}{k}\left[\left(k\log k - k\right) - (-1)\right]$

$= 1 - \frac{1}{k}.$

Thus $\lim_{k\to\infty} \log\left(\sqrt[k]{\frac{k^k}{k!}}\right) \leq 1.$

But



Thus, $\int_1^{k+1} \log(x)\,dx > \sum_{n=1}^{k} \log(k).$

Thus, $\log k - \frac{1}{k}\left(\sum_{n=1}^{k} \log(n)\right) > \log k - \frac{1}{k}\int_1^{k+1} \log(x)\,dx$

$= \log k - \frac{1}{k}\left(x\log(x) - x\right)\Big|_1^{k+1}$

$= \log k - \frac{1}{k}\Big(\big((k+1)\log(k+1) - (k+1)\big) - (-1)\Big)$

$= \log k - \frac{k+1}{k}\log(k+1) + \frac{k+1}{k} - \frac{1}{k}$

$= \log k - \log(k+1) - \frac{1}{k}\log(k+1) + 1 + \cancel{\frac{1}{k}} - \cancel{\frac{1}{k}}$

$= \log\left(\frac{k}{k+1}\right) - \frac{\log(k+1)}{k} + 1 \longrightarrow \log(1) - 0 + 1 = 1.$

Thus, $\lim_{k\to\infty} \log\left(\sqrt[k]{\frac{k^k}{k!}}\right) \geq 1.$

(~~Discuss~~ Discuss author's "$\sim$":)

Thus $\lim_{k\to\infty} \log\left(\sqrt[k]{\frac{k^k}{k!}}\right) = 1.$

Thus $\lim_{k\to\infty} \sqrt[k]{\frac{k^k}{k!}} = e.$ $\square$

II. For $0 < \lambda < 1$, we have $\limsup_{k\to\infty} \sqrt[k]{\sum_{p=k}^{\infty} \binom{p}{k} \lambda^p} < \infty.$

pf: Work through the book's proof.

Thm (26) ~~If the Taylor series of fo~~

If $f(x) = \sum_{k=0}^{\infty} c_k (x-x_0)^k$ has radius of convergence

R and $0<\sigma<R$, then $f(x)$ has bounded derivative growth rate (α) on $[x_0-\sigma, x_0+\sigma]$.

Pf Since $0<\sigma<R$, $\frac{\sigma}{R}<1$. $\exists \lambda \in (\frac{\sigma}{R}, 1)$.
Now $\sigma/R < \lambda$. $\exists N$ s.t. $k \geq N \Rightarrow |c_k|^{\frac{1}{k}} \leq \frac{1}{R}$.

Thus $|c_k|^{\frac{1}{k}}\sigma < \lambda \Rightarrow |c_k \sigma^k| < \lambda^k$.

Now, differentiate term-by-term to get

$$f^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)(k-2)\cdots(k-n+1)\, c_k (x-x_0)^{k-n}.$$

Thus,

$$|f^{(n)}(x)| \leq \sum_{k=n}^{\infty} k(k-1)\cdots(\quad)|c_k|\,|x-x_0|^{k-n}. \quad \star$$

$$k(k-1)\cdots(k-n+1) = \frac{k!}{(k-n)!} = n!\left(\frac{k!}{n!(k-n)!}\right) = n!\binom{k}{n}.$$

$|x-x_0| \leq \sigma$. Thus,

$$\star \leq n! \sum_{k=n}^{\infty} \binom{k}{n} |c_k|\,|\sigma|^{k-n} = \frac{n!}{\sigma^n} \sum \binom{k}{n} |c_k \sigma^k|$$

$$\leq \frac{n!}{\sigma^n} \sum_{k=n}^{\infty} \binom{k}{n} \lambda^k.$$

for $n \geq N$.

Thus,

$$M_n = \sup_{x \in [x_0-\sigma,\, x_0+\sigma]} \left| f^{(n)}(x) \right| \le \frac{n!}{\sigma^n} \sum_{k=n} \binom{k}{n} \lambda^k .$$

~~By III~~ Now

$$\sqrt[n]{\frac{M_n}{n!}} \le \frac{1}{\sigma} \sqrt[n]{\sum_{k=n}^{\infty} \binom{k}{n} \lambda^k} .$$

By II

$$\limsup_{n \to \infty} \frac{1}{\sigma} \sqrt[n]{\sum_{k=n}^{\infty} \binom{k}{n} \lambda^{1/k}} < \infty .$$

Thus

$$\alpha = \limsup \sqrt[n]{\frac{M_n}{n!}} < \infty,$$

and so f has bdd gber. growth rate on $[x_0-\sigma, x_0+\sigma]$. as claimed.

$\square$

27 **Analyticity Thm** A smooth function is analytic if and only if it has locally bdd der. growth rate.

Pf Thm 26 gives one direction. For the other, assume $f:(a,b)\to\mathbb{R}$ is smooth and has locally bdd α. Let $x_0\in(a,b)$ and let U be a nbhd of x_0 in (a,b) on which α is finite. Pick $\sigma>0$ s.t. $[x_0-\sigma, x_0+\sigma]\subset U$ and $\alpha\sigma<1$. By Thm 25, the Taylor series for f at x_0 converges on $[x_0-\sigma, x_0+\sigma]$. Thus, f is analytic.

Cor A smooth func with uniformly bdd derivatives is analytic.

Thm If $f(x) = \sum c_k(x-x_0)^k$ has radius of convergence R, then f is analytic on (x_0-R, x_0+R).

A brief detour into complex functions & series.

Def Let $\mathbb{C}$ be the complex plane (same metric as $\mathbb{R}^2$), and let $f: U \to \mathbb{C}$, $U \subset \mathbb{C}$. Assume $\exists \varepsilon > 0$ s.t. $B(z_0, \varepsilon) \subset \text{int}(U)$. Then the derivative of $f(z)$ at z_0 ~~exists~~ ~~and equals~~ is

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h},$$

if the limit exists. Note, h is a complex variable, $h \to 0$ means $\text{Re}(h), \text{Im}(h) \to 0$. The limit must be indep't of the path to $0 + i0$.

The derivative formulas you are used to still hold. Also, differentiability implies continuity. But it is not the same as having the $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ existing, $z = x + iy$. But we do have

Thm: The Cauchy-Riemann Conditions: let $w = f(z) = u(x,y) + iv(x,y)$ be diff'ble at $z_0 = x_0 + iy_0$. Then $u_x(x_0,y_0)$, $u_y(x_0,y_0)$, $v_x(x_0,y_0)$, $v_y(x_0,y_0)$ exist and

$$\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}\right\} \text{ Cauchy-Riemann equations.}$$

And these a sufficient.

Pf is elementary and short.

Thm If $f(z)$ is differentiable in some ε-nbhd of z_0 then: ~~the Taylor~~

$f(z)$ is infinitely diff'ble

The Taylor series converges to $f(z)$

That is, diff $\Rightarrow$ analytic!

The proof use line integrals. It also turns out that any diff'ble func has path independence!

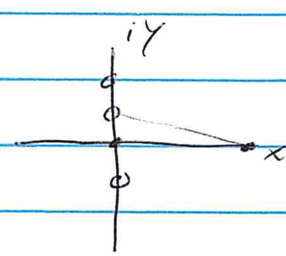
$$\oint_C f(z)\,dz = 0,$$

~~If f is diff'ble~~ (f must be diff inside & on C).

Also, there is a handy method of find the radius of convergence.

If f is analytic everywhere, $R = \infty$.
~~If~~ min the diff from z_0 to a point where f is not analytic is R.

Ex $f(z) = \dfrac{\sin(z)}{(z^2+1)(z-2i)}$



If $z_0 = 0$, $R = 1$.
If $z_0 = 7$, $R = \sqrt{49+1}$
If $z_0 = 5i$, $R = 3$.

This works if we restrict to $\mathbb{R}$.

In undergraduate textbooks on diff equations state thms like:

Thm If f and $\frac{\partial f}{\partial y}$ are conti in a rectangle R $|x| \le a$, $|y| \le b$, then $\exists$ a interval $(-h, h)$ $(0 < h \le a)$ in which $\exists !\, y = \phi(x)$ that solves

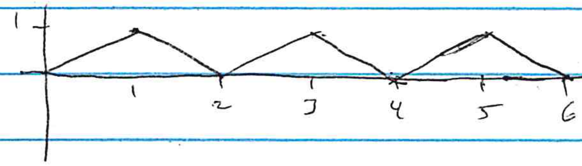
$$y' = f(x, y)$$
$$y(0) = 0.$$

Compare the proof they give to the proof of Picard's Thm. Are they basically the same?

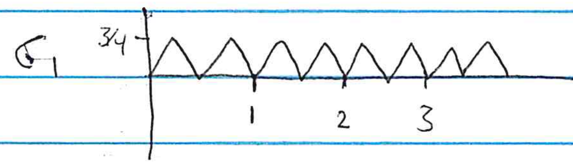
Section 7: Nowhere Differentiable Continuous Functions

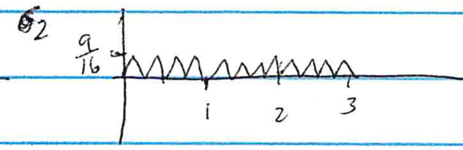
Thm $\exists\, f : \mathbb{R} \to \mathbb{R}$ that is continuous, but nowhere diff'ble.

Pf Let $\sigma_0(x) = \begin{cases} x - 2n & \text{if } 2n \le x \le 2n+1 \\ (2n+2) - x & \text{if } 2n+1 \le x \le 2n+2 \end{cases}$



Let $\sigma_k(x) = \left(\frac{3}{4}\right)^k \sigma_0(4^k x)$. Let $\Pi_k = \frac{2}{4^k}$ = period of σ_k.

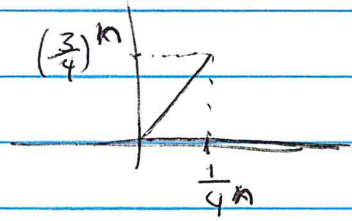




According to the Weierstrass M-test (pg 207), since $\|\sigma_k\| = \left(\frac{3}{4}\right)^k$ and $\sum \left(\frac{3}{4}\right)^k$ converges, $f(x) = \sum_{k=0}^{\infty} \sigma_k(x)$, ~~is well defined~~ convergence ~~and the~~ uniformly. Thus $f(x)$ is cont.

We claim $f'(x)$ d.n.e. $\forall x \in \mathbb{R}$.

The proof basically rests on the following: The slopes of the linear segments of $\sigma_n(x)$ is $\pm 3^n$.



Let $x \in \mathbb{R}$ and $\delta > 0$. Let $\delta_n = \frac{1}{2 \cdot 4^n}$. $\exists N \in \mathbb{N}$ s.t. $\delta_N < \delta$. Consider the quotient

$$S(\alpha) = \frac{f(x+\alpha) - f(x)}{\alpha}$$

for $\alpha \in (-\delta, \delta)$. We will show for any $n \geq N$ either $|S(\delta_n)|$ or $|S(-\delta_n)|$ is $\geq \frac{1}{2}(3^n+1)$.

Thus the limit $\lim_{\alpha \to 0} \frac{f(x+\alpha)-f(x)}{\alpha}$ d.n.e.

Now $S(\alpha) = \sum_{k=0}^{\infty} \frac{\sigma_k(x+\alpha) - \sigma_k(x)}{\alpha}$.

Suppose $\alpha = \pm \delta_n$. Then $|\alpha| = \delta_n = \frac{1}{2 \cdot 4^n} = 4^{k-(n+1)} \pi_k$.

Thus once $k > n$ $\sigma_k(x \pm \delta) = \sigma_k(x)$. Thus

$$S(\alpha) = \sum_{k=0}^{n} \frac{\sigma_k(x+\alpha) - \sigma_k(x)}{\alpha}.$$

Consider $k=n$, $\frac{\sigma_n(x \pm \delta_n) - \sigma_n(x)}{\pm \delta_n}$.

Either $[x-\delta_n, x]$ or $[x, x+\delta_n]$ ~~is in~~ has σ_n monotone. If the first, use $\alpha = -\delta_n$; if the second use $\alpha = \delta_n$. Then

$$\left| \frac{\sigma_n(x+\alpha) - \sigma_n(x)}{\alpha} \right| = 3^n.$$

For $0 \leq k < n$ we have

$$\left| \frac{\sigma_k(x \pm \delta_k) - \sigma_k(x)}{\pm \delta_k} \right| \leq 3^k.$$

Thus

$$|S(\alpha)| \geq 3^n - (3^{n-1} + 3^{n-2} + \cdots + 3 + 1) = 3^n - \left(\frac{3^n - 1}{3-1}\right) = \frac{1}{2}(3^n + 1).$$

Thus $f'(x)$ d.n.e. ∎

This result, by itself, is not all that surprising. What is, is that "most" cont. functions are like this! We need some definitions to make this precise. These will lead up to Baire's Thm.

Let X be a ~~metric~~ top. space.
For each $n \in \mathbb{N}$ let $G_n \subset X$ be open and dense.
Let $G = \bigcap G_n$. Sets constructed this way are sometimes called <u>residual sets</u>. We are interested in what conditions on X gaurentee that residual sets are them selves dense.
Spaces with this property are called <u>Baire space's</u>.

Suppose each point of X either has a certain property, or it does not. Then this property is said to be <u>generic</u> if every point in some residual set has it.

We will show that being nowhere differentiable is a generic property of $C^0([a,b], \mathbb{R})$.

Ex Let $\{q_1, q_2, \dots\}$ be an ordering of the rational numbers.
Let $G_n = \mathbb{R} - q_n$. Then each G_n is open and dense.
The intersection $\bigcap G_n$ is the set of irrational numbers.
Thus being irrational is a <u>generic</u> property.

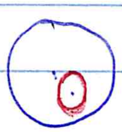
Baire's Thm Let M be a complete metric space, and let $\{G_k\}_{k=1}^{\infty}$ be a countable collection of open dense subsets of M. Then $\bigcap_{k=1}^{\infty} G_k = G$ is dense.

Pf (Royden's Real Analysis 3.)

Let U be an arbitrary open subset of M. We we show $U \cap G$ is nonempty, and hence $\overline{G} = M$. This will be done by constructing a seq $\{x_n\}$ whose limit x must be in $U \cap G$.

Let $x_1 \in G_1 \cap U$. Let B_1 be an open ball with radius r_1 centered at x_1 s.t. $B_1 \subset G_1 \cap U$.

Let $x_2 \in G_2 \cap B_1$ ($\neq \emptyset$ since G_2 is dense & B_1 is open)



Let B_2 = the open ball centered at x_2, with radius r_2 s.t.

$$B_2 \subset G_2, \quad r_2 < \frac{r_1}{2}, \quad r_2 < r_1 - d(x_1, x_2).$$

Then $\overline{B}_2 \subset B_1$. We continue inductively forming sequences $\{x_n\}$, $\{B_n\}$ s.t.

$$B_n \subset G_n, \quad \overline{B}_n \subset B_{n-1}, \quad \text{and } r_n \to 0.$$

We claim $\{x_n\}$ is Cauchy. To see this note that $m, n \geq N \Rightarrow x_n$ & $x_m \in B_N \Rightarrow d(x_n, x_m) < 2r_N$.

Since $r_N \to 0$, $\{x_n\}$ is Cauchy.

Since M is complete, $\exists\, x \in M$ s.t. $x_n \to x$.

Now we show that $x \in G$.

For $n > N$, $x_n \in B_{N+1}$ ~~and $\overline{B_{N+1}} \subset B_N \subset G_N$~~.
we have that $x \in \overline{B_{N+1}} \subset B_N \subset G_N$. Since
This holds for all N, $x \in G$. $\blacksquare$

Rmk The thm holds if the condition "complete metric space" is replaced with "compact Hausdorff space," which may not even have a metric. [Munkres Section 48]

Rmk The definition of a Baire space is equivalent to the following: Given any countable collection $\{A_n\}$ of closed sets in X with empty interiors, their union $\bigcup A_n$ also has empty interior. [Munkres, Section 48]

Other terminology.

residual $\Longleftrightarrow$ thick.

complement of

meager $\Longleftrightarrow$ thin.

A subset of the intersection of a ct'able collection of closed sets with empty interior, is of the "first category". If not, it is of the "second category".

Baire's Category Thm say's, in a Baire sp, no nonempty open subset is of the first category.

These ideas are related to the concepts of G_δ and F_σ sets.

$$G_\delta = \text{countable intersection of open sets.}$$

$$F_\sigma = \text{countable union of closed sets.}$$

Both come up in 501.

From Munkres § 49

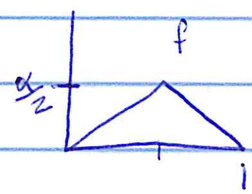
Def Let $f:[0,1]\to\mathbb{R}$ be cont. and let $h\in[0,\frac{1}{2}]$.
Define

$$\Delta_{f,h}(x) = \max\left\{\left|\frac{f(x+h)-f(x)}{h}\right|, \left|\frac{f(x-h)-f(x)}{-h}\right|\right\}$$

(at least one of which exists since $h\le\frac{1}{2}$.
Define

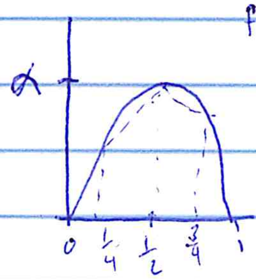
$$\Delta_{f,h} = \inf\{\Delta_{f,h}(x) \mid x\in[0,1]\}.$$

Ex



$\Delta_{f,\frac{1}{4}} = \alpha$. Study picture.

$f = 4\alpha x(1-x)$



$\Delta_{f,\frac{1}{4}} = \alpha.$

$$\frac{f(\frac{3}{4})-f(\frac{1}{2})}{\frac{1}{4}} =$$

$$\frac{\left(4\alpha\,\frac{3}{4}\cdot\frac{1}{4}\right)-\left(4\alpha\,\frac{1}{2}\,\frac{1}{2}\right)}{\frac{1}{4}}$$

$$\frac{\alpha\left(\frac{3}{4}-1\right)}{\frac{1}{4}} = -\alpha.$$

Thm Let $f:[0,1]\to\mathbb{R}$ be cont. $\forall \varepsilon>0$, $\exists\, g:[0,1]\to\mathbb{R}$ s.t. $\|f-g\|<\varepsilon$ and g is cont but nowhere diff'able.

Outline Pf $C_0^0([0,1],\mathbb{R})$ is a Baire sp. ~~Let $f\in C^0$~~

Define U_n to be all $f\in C^0$ s.t. for some $h\le \frac{1}{n}$, $\Delta_{f,h}>n$.

We will show,

Each U_n is open, dense. Then $U=\bigcap U_n$ is dense.
Then we show $\bigcap U_n$ consists of nowhere diff'able functions.

Un open Let $f\in U_n$. Let $0<h\le\frac{1}{n}$ be s.t. $\Delta_{f,h}>n$. Let $M=\Delta_{f,h}$. Let

$$\delta = h(M-n)/4.$$

Let $g\in C^0$ with $\|f-g\|<\delta$.

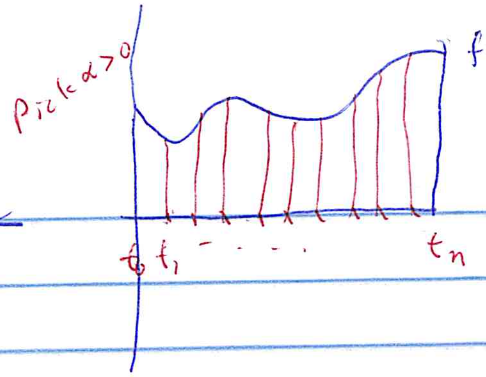
~~We claim~~ A calculation shows

$$\Delta_{g,h}(x)\ge \tfrac{1}{2}(M+n)>n \quad \forall x\in[0,1].$$

$h \le \frac{1}{n}$

Thus $g\in U_n$. Then U_n is open.

Un dense



over any $[t_i, t_{i+1}]$ $\max f - \min f) < \frac{\varepsilon}{4}$.

Pick $a_i \in (t_i, t_{i+1})$ s.t. if $f(t_i) \neq f(t_{i+1})$,

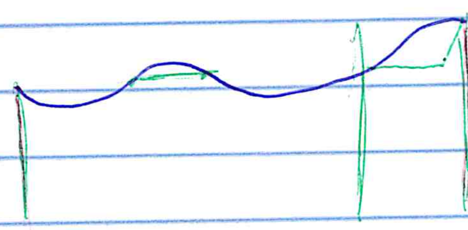
$$\frac{|f(t_{i+1}) - f(t_i)|}{|t_{i+1} - a_i|} > \alpha.$$

otw pick any a_i

Define $g^\alpha(x) = \begin{cases} f(t_i) & x \in [t_i, a_i] \\ f(t_i) + m_i\,(x - a_i) & x \in [a_i, t_{i+1}] \end{cases}$

where $m_i = \dfrac{f(t_{i+1}) - f(t_i)}{t_i - a_i}$

Now g^α is close to f but flat most places with $|\text{slope}| > \alpha$ elsewhere.



Now modify g^α on the flat areas as shown.

so all slopes are $> \alpha$, but $\|f - g\| < \varepsilon$.

$\cap U_n$ nowhere diff.

Let $f \in \cap U_n$.

$\forall n$, $\exists h_n$ s.t. $\Delta_{f,h_n}(x) > n$.

Then $\lim_{h \to 0} \Delta_{f,h}(x)$ d.n.e.

$\blacksquare$