Ch 6 : A nonrigorous intro to Lebesgue integration

What is wrong with Riemann integration?

Ex Let $f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q}. \end{cases}$

Upper integral is 1 and the low is 0.
Yet $f(x)=0$ only of a set of measure 0.
Thus $f(x)=1$ a.e. Shouldn't its integral be 1? Remember, R. int functions that differ only on a set a measure zero, have equal integrals. But $f(x)=1$ a.e. and is not R integrable.

An approach this problem is called Lebesgue integration. Basically we partition the y-axis

Let $D \subset \mathbb{R}$ be the disjoint union of $E_1 \cup E_2 \cup \cdots \cup E_n$.
Let $a_1, a_2 \ldots a_n$ be real numbers. Define

$$\varphi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}.$$

This is called a <u>simple function</u> (the def is more general)

Define $\displaystyle\int_D \varphi = \sum a_i \, \text{Size}(E_i)$.

Note that $\displaystyle\int_{[0,1]} \chi_{\mathbb{R}-\mathbb{Q}} = 1$, if $\text{size}(\mathbb{R}-\mathbb{Q}) = 1$.

It will turn out that limits of simple functions behave well under this integration and we can define the Lebesgue integral as a limit. But first, we have to answer, What is "size"? Ideally we want a mapping from subsets of $\mathbb{R}$ to $[0,\infty]$ s.t.

i. $m(E)$ is defined for each $E \subset \mathbb{R}$

ii $m([a,b]) = m((a,b)) = m([a,b)) = m((a,b]]) = b-a$.

iii if (E_n) is a seq. of disjoint subsets of $\mathbb{R}$ then $m(\bigcup E_n) = \sum m(E_n)$.

iv If $E \subset \mathbb{R}$ and $E_a^{*} = \{x+a \mid x \in E\}$, $a \in \mathbb{R}$, then $m(E) = m(E_a)$.

But, it is know that no such m exists! It is (i) that is dropped, or modified. That is, there always exist <u>nonmeasurable sets</u>.

Let $\mathcal{M} \subset$ all subsets of $\mathbb{R}$. We still want $\mathcal{M}$ to include all "reasonable" sets. This leads to the idea of a σ-algebra.

Def A collection of subset $\mathcal{A}$ of a set X is called a *σ-algebra* if

(i) $X \in \mathcal{A}$.

(ii) If $A \in \mathcal{A}$, then $X - A \in \mathcal{A}$. (Hence $\emptyset \in \mathcal{A}$.)

(iii) If $A_n \in \mathcal{A}$, $n = 1, 2, \ldots$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

(It follows that $\bigcap A_n \in \mathcal{A}$ as well.)

(Compare/contrast to def. of a topology.)

If $\mathfrak{M}$ is a σ-alg. for X, then the pair $(X, \mathfrak{M})$ is called a *measurable space*.

If $f : X \to Y$, where Y is a top. sp., and $f^{-1}(V) \in \mathfrak{M}$ $\forall$ open $V \subset Y$, we say f is a *measurable function*.

Fact: If $E \in \mathfrak{M}$, then χ_E is *measurable*.

Fact If $\mathcal{F}$ is any ~~subset~~ family of subsets of X, $\exists$ a smallest σ-alg that contains $\mathcal{F}$.

Def Let $(X, \mathcal{T})$ be a top. sp. The smallest σ-alg. containing $\mathcal{T}$ are called the *Borel sets* of $(X, \mathcal{T})$.

Note that G_δ and F_σ sets are Borel sets.

The Borel sets of $\mathbb{R}$, denoted, $\mathcal{B}$, are all be the ones we can measure.

How to define our measure? First we define outer measure.

Let $A \subset \mathbb{R}$. Let ~~...~~ $\mathcal{C}$ to the collection of all countable collections of open intervals that cover A. $\{I_n\} \in \mathcal{C}$ mean $A \subset \cup I_n$. The <u>outer measure</u> of A is

$$m^* A = \inf_{\{I_n\} \in \mathcal{C}} \sum \ell(I_n), \quad \ell(I_n) = \text{length}.$$

(∞ is allowed)

<u>Facts</u> $m^* \phi = 0$.

$A \subset B \Rightarrow m^* A \leq m^* B$.

$m^*([a,b]) = b-a$, etc. (But the proof is not trivial!)

Let (A_n) be a countable collection of subsets of $\mathbb{R}$. Then

$$m^*(\cup A_n) \leq \sum m^* A_n$$

(Note: you can't use induction)

If A is countable, $m^* A = 0$.

However, m^* is not countably additive! We don't have

$$m^*(\cup E_n) = \sum m^* E_n$$

for disjoint E_n. The proof involves ~~the construction of nonmeasureable sets,~~ in difficult.

Here is how to get around this.

<u>Def</u> A set $E \subset \mathbb{R}$ is <u>Lebesgue measurable</u> if

$\forall A \subset \mathbb{R}$ we have $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$.

Let $\mathcal{M}$ denote the ~~set~~ collection of measurable subsets of $\mathbb{R}$.
Then $m : \mathcal{M} \to [0, \infty]$, is $m(E) = m^*(E)$, and
m is a measure.

<u>Facts</u> $\mathcal{M}$ is a σ-alg. $\mathcal{B} \subset \mathcal{M}$.

If $m^* E = 0$, then $E \in \mathcal{M}$.

<u>Thm</u> For $E \subset \mathbb{R}$ the following are equivalent
(a) E is measurable
(b) $\forall \varepsilon > 0$ $\exists$ open set O, $E \subset O$, $m^*(O \setminus E) < \varepsilon$.
(c) $\forall \varepsilon > 0$ $\exists$ closed set F, $F \subset E$, $m^*(E \setminus F) < \varepsilon$.
(d) $\exists$ G_δ set G, $E \subset G$, $m^*(G - E) = 0$
$\exists$ F_σ set F $F \subset E$, $m^*(E - F) = 0$.

Measurable Functions.

Def Let X be any set and $\mathcal{M}$ ~~any~~ a σ-alg for some measure. Let Y be a top. sp. Then $f: X \to Y$ is measurable if $f^{-1}(\text{open})$ is a measurable set, $\in \mathcal{M}$.

Thm Let $D \subset \mathbb{R}$ be measurable. Let $f: D \to \mathbb{R}$. Then, if f is measurable all the classes of sets below are measurable, and if any one class is measurable, f is.

$f^{-1}((\alpha, \infty))$ $\qquad$ $f^{-1}((\alpha, \beta))$

$f^{-1}([\alpha, \infty))$ $\qquad$ $f^{-1}([\alpha, \beta))$

$f^{-1}((-\infty, \alpha))$ $\qquad$ $f^{-1}([\alpha, \beta])$

$f^{-1}((-\infty, \alpha])$ $\qquad$ $f^{-1}([\alpha, \beta])$

Thm If f & g are measurable, and $c \in \mathbb{R}$, then $c \in \mathcal{M}$, $f+g \in \mathcal{M}$, $fg \in \mathcal{M}$.

Thm If $f \in$ measurable and $f = g$ a.e., $g \in$ measurable.

Thm If $E \in \mathcal{M}$, χ_E is measurable.

Recall the simple functions: $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$.

Fact Let f be bdd on $E \in \mathcal{M}$, $mE < \infty$. In order that

$$\inf_{f \le \psi} \int_E \psi = \sup_{f \ge \varphi} \int_E \varphi$$

when the inf & sup are over all ψ & φ simple, it is n. & s. that f be measurable.

Def If f is a bdd measurable func. on $E \in \mathcal{M}$, $mE < \infty$,

$$\int_E f = \inf_{\psi \ge f} \int_E \psi$$

This is called the Lebesgue int. of f.

Prop Let $f:[a,b] \to \mathbb{R}$ be bdd. If $f \in \mathcal{R}$, then f is meas. and

$$R\int_a^b f(x)\,dx = L\int_a^b f(x)\,dx$$

Prop Let $\{f_n\}$ be a seq of measur. func. on $E \in \mathcal{M}$, $mE < \infty$. If $f(x) = \lim f_n(x)$, $x \in E$, then

$$\int f = \lim \int f_n.$$

We did not have this before.

Ex Let $\{q_1, \dots\} = \mathbb{Q} \cap [0,1]$. Let $f_n(x) = \sum_{i=1}^{n} \chi_{\{q_i\}}$.

Then $f_n \to \chi_{\mathbb{Q} \cap [0,1]}$.

$$L\int f_n = R\int f_n = 0. \quad R\int f \text{ d.n.e.} \quad L\int f = 0.$$

$\boxed{\text{Now we can push beyond bdd functions.}}$

Let f be non neg meas. func. on $E \in \mathcal{M}$. Define
↳ need not be finite.

$$\int_E f = \sup_{h \le f} \int_E h$$

where h is bdd meas. func. s.t. $m(x \mid h(x) \neq 0) < \infty$.

Facts $\int_E cf = c\int_E f \qquad c > 0$

$\int_E f+g = \int_E f + \int_E g$

$f \le g$ a.e. $\int_E f \le \int_E g$.

Def A non neg measurable f is integrable over $E \in \mathcal{M}$ if

$$\int_E f < \infty.$$

Fact $f \geq 0$, $\int_E f = 0 \Rightarrow f = 0$ a.e.

We generalize to ~~no~~ +/− functions.

For $f: \cancel{\mathbb{R}}\, E \to \mathbb{R}$, Let $f^+(x) = \max\{f(x), 0\}$

$f^-(x) = \max\{-f(x), 0\}$.

$$f = f^+ - f^-.$$

Def: $$\int_E f = \int_E f^+ - \int_E f^-.$$

$\underline{\text{DCT}}$
LCT

Let $E \in \mathcal{M}$, g nonneg int. over E.
Let $\{f_n\}$ be a seq of m. funcs s.t. $|f_n| \le g$ on E ~~for a.e. x ∈ E~~.
Let $f(x) = \lim f_n(x)$ a.e. on E. Then

$$\int_E f = \lim \int_E f_n.$$

[Compare to Fatou's Lemma & the MCT.]

The FTC takes the following form.

Def Let $g:[a,b]\to\mathbb{R}$. Let g is <u>absolutely continuous</u> if $\forall \varepsilon>0$, $\exists \delta>0$ s.t.

Whenever $(\alpha_1,\beta_1),\dots(\alpha_n,\beta_n)$ are disjoint intervals in $[a,b]$ we have

$$\sum_{i=1}^{n} \beta_i-\alpha_i<\delta \Rightarrow \sum_{i=1}^{n}\left|g(\beta_i)-g(\alpha_i)\right|<\varepsilon.$$

Cont $\overset{\Leftarrow}{\not\Rightarrow}$ A.C.

Thm (FTC) Let $f:[a,b]\to\mathbb{R}$ be Lebs int.iable. Then

(a) $F(x)=\int_a^x f(t)\,dt$ is abs. cont.

(b) $F'(x)$ exist and $=f(x)$ for a.e. $x\in[a,b]$.

(c) If $G(x)$ is abs. cont. $G'(x)$ exists$^{a.e.}$ and if $=f(x)$ a.e., Then $G=F+C$. ~~(a.e.)~~

Finally

Read Littlewood's Three Principles in App. D or 3.6 of Royden.

Fact: F is an int $\int_a^x f$ iff it is abs. cont.

Little woods ________.

The L^p Spaces

$p>0$. $L^p = L^p([0,1]) = \{ f \in \mathcal{M} \mid \int_0^1 |f(x)|^p < \infty \}$

It is easy to show L^p is a vector space.

It is given the norm $\|f\|_p = \sqrt[p]{\int_0^1 |f|^p}$.

L^∞ = all a.e. bdd measurable functions on $[0,1]$

$$\|f\|_\infty = \inf\{M : m\{x : f(x) > M\} = 0\}$$

$p \geq 1$: $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

$\forall p$ $\|\alpha f\|_p = |\alpha| \|f\|_p$

$$\|f\|_p = 0 \Leftrightarrow f = 0 \text{ a.e.}$$

If we let $f \sim g$ if $f = g$ a.e.

We have for $p \geq 1$, L^p is a normed linear space.

A complete normed linear space is called a Banach Space
A " inner prod. lin space is called a Hilbert space

~~$1 \le p \le \infty$~~

Minkowski:

$0 < p < 1$: $\|f+g\|_p \geq \|f\|_p + \|g\|_p$

$1 \le p \le \infty$ $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

Hölder $1 \le p, q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

$f \in L^p$, $g \in L^q$. Then $f \cdot g \in L^1$ & $\int |fg| \le \|f\|_p \|g\|_q$

Linear functionals $F : X \to \mathbb{R}$ $F(\alpha f + \beta g) = \alpha F(f) + \beta F(g)$

(X normed v. sp.)

bdd if $|F(f)| \le M \|f\|$.

Then define $\|F\| = \sup\limits_{\substack{f \in X \\ \|f\| \neq 0}} \frac{|F(f)|}{\|f\|}$.

$\forall g \in L^q(E)$, define $G : L^p_{(E)} \to \mathbb{R}$ by $G(f) = \int_E fg$

Then $\|G\| = \|g\|_q$

Riesz Rep. Thm: Let $G : L^p \to \mathbb{R}$ be a bdd linear functional, $1 \le p < \infty$. $\exists!\ g \in L^q$ s.t.

$$G(f) = \int fg$$

and $\|G\| = \|g\|_q$.

Ergodic Thy

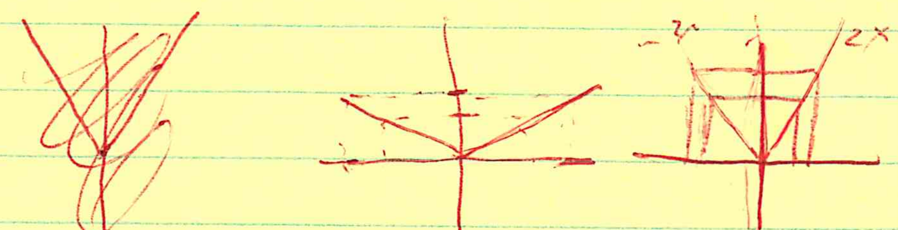
<u>Def</u> Let $(X, \mathcal{M}, m)$ be a set X, with σ-alg $\mathcal{M}$ and measure m. If $m(X)=1$, we call this ~~this~~ triple a probability space.

<u>Def</u> Let $X = G$ be a compact top. group. Let $\mathcal{B}(G)$ be the σ-alg of Borel subset of G, ~~that are translation invariant~~ Then m, under some conditions, is called Haar measure.

The most common examples are S^1 & tori, T^n, under rotations.

Def Let $T : X \to X$. If $m(T^{-1}(B)) = m(B)$ $\forall B \in \mathcal{M}$, we say T is <u>measure preserving</u>.

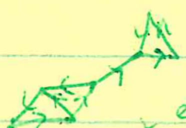
<u>Ex</u> $T(x) = x$.



$T(X) = |2X|$

<u>Def</u> A meas pres $T : X \to X$ is <u>ergodic</u> if the only $B \in \mathcal{M}$ s.t. $T^{-1}(B) = B$ have $m(B) = 0$ or $m(B) = 1$.

Think mixing pot.



Not ergodic
no dense orbit.

Thm The following are equivalent

ⓐ T is ergodic

ⓑ $\forall\, A \in \mathcal{B}$ with $m(A) > 0$ we have $m\left(\bigcup_{n=1}^{\infty} T^{-n} A\right) = 1$.

ⓒ $\forall\, A, B \in \mathcal{B}$ with $m(A) > 0$, $m(B) > 0$ $\exists\, n > 0$ with $m(T^{-n} A \cap B) > 0$.

Let $L^p(X, \mathcal{M}, m)$ = all measurable functions $f : X \to \mathbb{C}$ / $\mathbb{R}$ s.t. $\int_X |f|^p < \infty$.

Then $T : X \to X$ induces a map $U_T : L^p \to L^p$ as follows.

$$U_T(f)(x) = f(Tx).$$

Ex Rotations ~~are~~ of unit circle are ergodic iff they are irrational.

Thm Let $f \in L^1$. ~~Def~~ and $f_n = \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x))$.

Then $f_n \xrightarrow{a.e} f^* \in L^1$ and, $f^* \circ T = f^*$ a.e. and

$$\int f^* = \int f.$$

If T is ergodic, f^* is a constant and

$$\lim \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \int f \quad \text{a.e.}$$

Thm For almost all numbers in $[0,1)$ the frequency of 1's in the binary expansion is $\frac{1}{2}$.

Pf Let $T:[0,1)\to[0,1)$ be $T(x) = 2x \bmod 1$. It is ergodic.

Let Y denote the set of pts of $[0,1)$ that have a unique binary expansion. $[0,1)\setminus Y$ is countable so $m(Y)=1$.

Let $x\in Y$ and $x = a_1/2 + a_2/4 + \cdots \qquad a_i\in\{0,1\}$.

Then

$$T(x) = \frac{a_2}{2} + \frac{a_3}{2^2} + \cdots$$

Let $f(x) = \chi_{[\frac{1}{2},1)}(x)$. Then

$$f(T^i(x)) = f\left(\frac{a_{i+1}}{2} + \frac{a_{i+2}}{2^2} + \cdots\right) = \begin{cases} 1 & \text{iff } a_{i+1}=1 \\ 0 & \text{iff } a_{i+1}=0. \end{cases}$$

For $x\in Y$, the number of 1's in the first n digits

is $\sum_{i=0}^{n-1} f(T^i(x))$. But

$$\frac{1}{n}\sum_{i=0}^{n-1} f(T^i(x)) \xrightarrow{a.e.} \int \chi_{[\frac{1}{2},1)} = \frac{1}{2} \qquad \blacksquare$$