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Berkeley's Attack on the Infinitesimal

George Berkeley was an 18th century philosopher who critiqued certain elements of the mathematical thought of his time. He particularly dismissed the developments of Newtonian calculus, viewing it as failing mathematical rigor because of its reliance on infinitesimals. The notion of an arbitrarily small quantity baffled Berkeley and led him to publish *The Analyst* in which he criticized the foundations of Newtonian analysis, a work that would credit Berkeley with the rejection of calculus. Berkeley questioned the means by which "Mathematicians unlock the secrets of Geometry" and made it his endeavor in *The Analyst* to show "whether this Method be clear or obscure, consistent or repugnant, demonstrative or precarious . . . with the utmost impartiality" (*Analyst* §3). It was his philosophical way of perceiving reality that led Berkeley to this argument, for he "was determined to hold on to the idea that we are directly aware of the physical world itself, whilst accepting that what we are aware of must be mind-dependent ideas. He was, therefore, forced to conclude that the physical world consists essentially of ideas in our minds– that its *esse is percipi*: for material objects, *to be* is *to be perceived*" (*Principles* xiii-xiv).

This description of Berkeleyan thought is fundamental for understanding his debate against Newtonian analysis because for Berkeley, his inability to perceive the concept of infinitesimals was the essential reason why he discounted calculus. Thus, Berkeley thought of the infinitesimal as something that did not exist in the physical world. His lack of understanding infinitely small magnitudes enabled Berkeley to challenge Newton's mathematical endeavors relating to fluxions. Fluxions, in Newtonian mathematics, are applied in problems that changed continuously; they are the fundamental infinitesimal quantity that compose modern day calculus (Jesseph 144). However, Berkeley's argument against Newtonian analysis is not purely based on his personal philosophical conflicts but is also founded on mathematical examples in which he finds "Emptiness, Darkness, and Confusion . . . direct Impossibilities and Contradictions" (*Analyst* §8). In *The Analyst* Berkeley demonstrates that the notion of the infinitesimal is intangible and nonsensical, that Newton's fluxion is indeed an infinitesimal quantity and that the proofs conducted by means of this fluxion are invalid and contradictory, and that the correct results obtained in calculus by Newton and other mathematicians were merely a consequence of a compensation of errors.

The critique of calculus given by Berkeley relies heavily on the concept of the infinitesimal; therefore, it is necessary to provide a thorough description of how this was perceived in the 18th century. The following is an example that illustrates the notion of a quantity that is infinitely small. By constructing a circle out of small segments it is possible to view the area of the circle as a sum of the areas of an infinite number of arbitrarily small isosceles triangles. These triangles have bases of infinitely small lengths which provide a reasonable example of the term infinitesimal for: "the admission of a zero length would force us to treat the circumference of the circle as an infinite sum of the form 0 + 0 + 0 + . . . , which is equal to zero" (Jesseph 130). Now, if these bases were measurable by a positive real number then the circle could not possibly be composed of isosceles triangles because there would always be a portion of the of the area left out. Therefore, the bases of these triangles must not be zero and they must not be measurable by any positive real number, thus they are viewed in mathematics as having an arbitrarily small length. "We are led to regard infinitesimals as quantities greater than zero but less than any positive real number" (Jesseph 130). This is the concept of the infinitesimal that Berkeley was familiar with and the one that he criticized as intangible in his exploration of calculus.

Berkeley's confusion about the infinitesimal led him to regard it as an object which deserved no symbol. He states: "Tis plain to me we ought to use no sign without an idea answering to it; and tis as plain that we have no idea of a line infinitely small, nay, tis evidently impossible there should be any such thing, for every line, how minute soever, is still divisible into parts less than itself; therefore there can be no such thing as a line *quavis data minor* or infinitely small" (*Works* 4:235-236). This was an important statement, for Berkeley's argument because he seemed to believe that mathematicians such as Newton seemingly blindly trusted the symbols which represented infinitesimals in their proofs of geometrical theorems. For Berkeley, that which could not be understood as having any real meaning had no role in mathematical rigor. In his essay "Of Infinites", he argues that "because the notion of an infinitely small magnitude is not clearly conceivable, the introduction of infinitesimals violates a criterion of rigor which demands that the objects of mathematical investigation be clearly conceived" (Jesseph 163-164). The notion that infinitely small quantities bear no meaning whatsoever is portrayed in Berkeley's pamphlet *The Analyst* in which he described why these quantities were so puzzling to him. His argument firstly dealt with Newton's fluxion, which Berkeley viewed in the same manner as the infinitesimal. "These Fluxions are said to be nearly as the Increments of the flowing Quantities, generated in the least equal Particles of time" (*Analyst* §3). It is Berkeley's inability to grasp the concept of these fluxions which leads him to discredit Newton's calculus which is based upon them. Berkeley even admits that the idea of an infinitely small quantity is above his ability: "Now to conceive a Quantity infinitely small, that is, infinitely less than any sensible or imaginable Quantity, or than any the least finite Magnitude, is, I confess, above my Capacity" (*Analyst* §5).

Much of Berkeley's argument against calculus relies on his ineptitude when it came to the understanding of things infinitely small and even more so by the fact that he considered his reasoning abilities and intuition on the term to be complete. He therefore assumed that it was above the ability of any man and thus must be wrong: "But to conceive a Part of such infinitely small Quantity, that shall be still infinitely less than it, and consequently though multiply'd infinitely shall never equal the minutest finite Quantity, is, I suspect, an infinite Difficulty to any Man whatsoever" (*Analyst* §5). Berkeley exhausted his philosophical notion that infinitesimals have no meaning in the physical world and hence have no place mathematics by explaining his befuddled notion of these quantities: "That is, they (mathematicians) consider Quantities infinitely less than the least discernible Quantity; and others infinitely less than those infinitely small ones; and still others infinitely less than the preceding Infinitesimals, and so on without end or limit" (*Analyst* §6). He continues on his bewildered path regarding arbitrarily small measures to be baffled by the notion of adding "a Million of Millions of these Infinitesimals" to a quantity and having the quantity "be never the bigger" (*Analyst* §6). It was this foundation, the notion of the infinitesimal as an unintelligible quantity, that Berkeley built his argument against calculus upon; it was an argument that revolved around his interpretation of an infinitely small quantity that led him to his mathematical rebuttals of Newtonian calculus.

Berkeley was not a mathematician; however, he pursued mathematics from both a philosophical level as well as a mathematical one. He attempted to justify his claim that calculus was based upon a fictitious quantity, the infinitesimal. In the case of Newtonian calculus Berkeley argued that the fluxion was misused in proofs of geometrical theorems and examples. In *The Analyst* Berkeley critiques “two Newtonian proofs of elementary theorems in the calculus, the first of which is a method for finding the fluxion of a product of two flowing quantities, Newton’s version of the ‘product rule’ for differentiation of a product” (Jesseph 190). In modern calculus this is equivalent to the derivative of the product of two functions _(x) and g(x) which is equal to __(x)g(x) + _(x)g_(x). Newton showed this product by using a rectangle which had sides A and B that were considered flowing quantities, with moments a and b. The proof that Newton provided first considered the case in which each of the flowing quantities A and B was without one-half of its moment, this yielded a rectangle with an area of

$$(A - a/2) \times (B - b/2).$$

Now, multiplying these two quantities together yields equation (1)

$$AB - (1/2)aB - (1/2)bA + (1/4)ab.$$

The second case that Newton considered was the one in which A and B were increased by the remaining half of their moments a and b, it follows that:

$$(A + a/2) \times (B + b/2).$$

This, multiplied through yields equation (2)

$$AB + (1/2)aB + (1/2)bA + (1/4)ab.$$

Newton’s claim was that the “moment of the product will be the difference” between equations (1) and (2) which was merely

$$aB+bA.$$

Berkeley claimed that the result reached by Newton was false and to obtain the true increment of the area of the rectangle one need to simply compare the area

AB to the product $(A + a) \times (B + b)$.

Leaving a difference in areas equal to

$$aB + bA + ab. \qquad \text{(\textit{Analyst} §9)}$$

Berkeley's answer was different from Newton's result by the additional term ab. "Berkeley astutely reveals a fundamental flaw in Newton's procedure" that "depends upon the confusing supposition that we can divide momentary increments of negligible magnitude into parts" (Jesseph 190-191). Berkeley contends that "no matter how we interpret the doctrine of moments, Newton's procedure requires the use of infinitely small quantities and his denial of using infinitesimal is simply sophistical" (Jesseph 191). Berkeley was convinced that the term ab must be removed from this case in order for the result to make any sense. He concludes that "notwithstanding all this address and skill the point of getting rid of ab cannot be obtained by legitimate reasoning" (*Analyst* §10).

Berkeley thought of mathematicians such as Newton as purely "Men accustomed rather to compute than to think; earnest rather to go on fast and far, than solicitous to set out warily and see their way distinctly" (*Analyst* §10). He viewed himself as a very careful critic of men who were, in a sense, sloppy in their calculations, and men who were more inclined to care about the results reached by their calculations rather than the means they used to reach them. However, in this case it was not Berkeley's genius insight into the problem of moments that allowed him to find a flaw with the Newtonian method of computing the moment of a product but rather that Newton was incomplete in the explanation of his proof: "Nor is it surprising that, for all his profound intuitive grasp of the rightness and good sense of what he was doing, Newton was unable to give complete logical validity to an infinitesimal calculus, leaving cracks to which Berkeley and other critics could insert powerful and destructive wedges" (Hall 32).

Newton apparently avoided viewing his moments as infinitesimals. It was this move by Newton that motivated Berkeley to see calculus as not fulfilling mathematical rigor. Berkeley insists in section 11 of *The Analyst* that Newton's use of moments is indeed equivalent to the use of infinitesimals: "Certainly, Newton's mysterious procedure is motivated by a desire to avoid embarrassing questions about infinitesimal magnitudes, but in setting out a proof of this sort Newton has instead shown how unrigorous the calculus really is" (Jesseph 192). In this passage Berkeley illustrates his complaints against calculus by demonstrating that Newton's moments are indeed infinitely small quantities; therefore, showing that Newton's development of calculus was not complete and failed mathematical rigor. Berkeley states the following:

> If by a Momentum you mean more than the very initial limit, it must be either a finite Quantity or an Infinitesimal. But all finite Quantities are expressly excluded from the notion of a Momentum. Therefore the Momentum must be an Infinitesimal. And indeed, though much Artifice hath been employ'd to escape or avoid the admission of Quantities infinitely small, yet it seems ineffectual. For ought I see, you can admit no Quantity as a Medium between a finite Quantity and nothing, without admitting infinitesimal. An Increment generated in a finite Particle of Time, is it self a finite Particle; and cannot therefore be a Momentum. You must therefore take an Infinitesimal Part of Time wherein to generate your Momentum (*Analyst* §11).

Berkeley here has demonstrated by both mathematical and philosophical means that there do indeed exist flaws in the calculus brought forth by Newton.

Berkeley's next attack on the Newtonian methods of analysis were against yet another proof given by Newton in which he furthered the previous proof to find the fluxion of any power of a flowing quantity. Berkeley's concern and hence his dismissal of this proof relies on the following lemma which he states in section 12 of *The Analyst:*

> If with a view to demonstrate any Proposition, a certain Point is supposed, by virtue of which certain other Points are attained; and such supposed Point be it self afterwards destroyed or rejected by a contrary Supposition; in that case, all the other Points, attained thereby and consequent thereupon, must also be destroyed and rejected, so as from thence forward to be no more supposed or applied in the Demonstration.

This lemma is purely stating that contradictory assertions are not to be permitted in a demonstration; to admit such contradictory elements would negate the demonstration altogether. It is this lemma that supports Berkeley's next argument against the Newtonian method for finding the fluxion of a power of a flowing quantity. Newton's proof, which taken directly from the "Introduction" to Newton's *Quadrature of Curves*, is given directly in *The Analyst* as follows:

> I suppose that the Quantity x flows, and by flowing is increased, and its Increment I call o, so that by flowing it becomes $x + o$. And as x increaseth, it follows that every Power of x is likewise increased in a due Proportion. Therefore as x becomes $x + o$, x^n will become $(x+o)^n$: that is, according to the Method of Infinite Series
>
> $$x^n + nox^{n-1} + \frac{(n^2 - n)}{2}o^2x^{n-2} + \&c.,$$
>
> And if from the two augmented Quantities we subduct the Rood and the Power

respectively, we shall have remaining the two Increments, to wit,

$$o \text{ and } nox^{n-1} + \frac{(n^2 - n)}{2}o^2x^{n-2} + \&c.,$$

which Increments, being both divided by the common Divisor o, yield the Quotients

$$1 \text{ to } nx^{n-1} + \frac{(n^2 - n)}{2}ox^{n-2} + \&c.$$

which are therefore Exponents of the Ratio of the Increments. Hitherto I have supposed that x flows, that x hath a real Increment, that o is something. And I have proceeded all along on that Supposition, without which I should not have been able to have made so much as one single Step (*Analyst* §14).

Berkeley's argument here takes the premise of the previously stated lemma in that he surmises that Newton made contradictions regarding the quantity o. In this Newtonian proof Berkeley points out that Newton made the assumption that the increment o is initially treated as a positive quantity and then after the simplification of the ratios by dividing out o it is treated as having a quantity equal to zero. He then states, using the lemma as backing, that "when once the second Supposition or Assumption is made, in the same instant the former Assumption and all that you got by it is destroyed, and goes out together" (*Analyst* §16). Therefore the Newtonian proof for deriving the fluxion of any power of a flowing quantity was proven by Berkeley to have no grounds as a mathematical derivation or proof. Berkeley found yet another example of Newtonian analysis by which he furthered his claim that calculus was based upon unrigorous and sometimes completely false proofs due to the intangible quantity, the infinitesimal.

Berkeley made these attacks against Newton's use of the fluxion by both showing that the fluxion as an infinitesimal quantity had no philosophical meaning as well as having no place in mathematical proofs. However, Berkeley was intrigued by the fact that Newton obtained the correct results when he solved geometrical problems using his method of fluxions. As Berkeley states, “I have no Controversy about your Conclusions, but only about your Logic and Method” (*Analyst* §20). He was therefore not arguing that the results achieved by Newton were incorrect but that the means by which he achieved those results were questionable, and, in Berkeley’s opinion wrong. This inspired Berkeley to produce a thesis he called *The Compensation of Errors* in which he showed how the incorrect reasoning of calculus was still able to generate correct results. He wanted to show how correct results were achieved by false means. Berkeley states clearly the intention of his *Compensation of Errors* thesis as follows: “forasmuch as it may perhaps seem an unaccountable Paradox, that Mathematicians should deduce true Propositions from false Principles, be right in the Conclusions and yet err in the Premises; I shall endeavour particularly to explain why this may come to pass, and shew how Error may bring forth Truth, though it cannot bring forth Science” (*Analyst* §20). Berkeley shows in a very detailed example that the infinitesimal quantities cancel each other when left in the problems. “The explanation of the correct results given by Berkeley . . . was thus that the ignored quantities if restored would cancel” (Wisdom 23-24). “To explain this he produced an ingenious thesis that there is a compensation of errors, that is, the *one* error introduced into the incrementary ratio is compensated by *one* error in the expression of geometrical properties in terms of infinitesimals” (Wisdom 23). With this thesis Berkeley was able to show how mathematicians that used calculus, such as Newton, were able to render correct results despite the fact that they used quantities such as the infinitesimal. Berkeley’s argument against calculus in essence ends with his *Compensation of Errors* thesis.

In *The Analyst* Berkeley criticizes and attacks the logic and methods behind Newton’s fluxion, or infinitesimal calculus. He showed that contrary to what Newton stated, the Newtonian fluxion was actually an infinitesimal quantity. He also showed that the proofs conducted by means of the fluxion held no esteem in the rigor necessary for mathematical reason and that the results achieved in calculus by Newton and other mathematicians were only correct due to a compensation of errors. "The ensuing controversy in which Berkeley joined was violent and prolonged. Mathematicians were ranged on both sides . . . The problem was resolved, so far as Berkeley's criticism was concerned, in 1821 by Cauchy's theory of limits". However, Berkeley was not proven incorrect in his argument against calculus: "The outcome is that Berkeley has been proved correct in his criticism: the concept of the infinitesimal had to be eliminated from the theory". Notably, Newton's calculus was not incorrect either, it just required the detailed application of Cauchy's theory of limits (Wisdom 23). Therefore, Berkeley, while not a mathematician, did have the necessary philosophical insight to dive into the mathematics of his time and extract contradictions and seeming impossibilities. It is very interesting to consider how the notion of a concept, such as the infinitesimal, as having no grounds in reality can shape the history of mathematics and question the theories generated by Newton, who was arguably the best scientific thinker in the history of the world.

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