Math 452
Real Analysis
Prof. Sullivan
Fall 2016

Goals: Develop a rigorous understanding of the real number line, convergence in metric spaces, the formal basis of calculus (review), and uniform convergence. Develop a beginning understanding of function spaces and the measure of subsets of the reals as preparation for a rigorous course in Lebesgue measure theory and integration as well as preparation for applied courses in approximation theory and physics courses that make use of function spaces.

Students will further improve their theorem proving abilities. All proofs are to be written in grammatically correct sentences.

The textbook, see below, is meant to be read carefully. You cannot just skim through for the important parts like with many lower level textbooks. You should read all the exersices not just the asssigned ones. This is holds for most any upper level textbook.

Textbook: *Real Mathematical Analysis*, 2nd Edition, by Charles Pugh, Springer. You can get it from: Amazon, Springer, Powell's, etc.

Note: The 2nd edition is not that different from the 1st. So, you can use the 1st edition if you like.

Grading: Weekly homework, a midterm, and a final, will each count as one third of the final grade.

Warning: I know solutions can be found on line. I have copies. Do not use them. Do not look at them. The point is to learn how to think for yourself. If you are copying I will ask that you drop the course and I will not grade your work.

Lectures online: Lecture notes and the lectures themselves will be available online. Off campus students will use these instead of coming to class. On campus students can use them if they wish.

Lectures and Lecture Notes

Homework Assignments The dates are from 2014; I'll update this soon.

- Set 1. hwk1.pdf. It is due the first day of class.
- Set 2. Chapter 1: 16abc, 39, 40 (bonus). Due Monday, August 25.
- Set 3. Chapter 1: 42ab, 44a, 44b (bouns). Due Wednesday, Sept 3.
- Set 4. Chapter 2: 5, 6, 8, 14. Due Monday, Sept 8.
- Set 5. Chapter 2: 17, 22, 23, 26. Due Monday, Sept 15.
- Set 6. Chapter 2: 28, 30a, 39, 41. Due Monday, Sept 22.
- Set 7. Chapter 2: 34a (prove or give a counter example), 37 (bonus), 40abcd, 74 ab, 130 (bonus!). Due Monday, Sept 29.
- Set 8. Chapter 3: 1,2abc, 17. Due Monday, Oct 6.
- Bonus Set! Chapter 3: Present 70 to the class.
- Midterm, two hours. This will be in the evening, sometime in the week of Oct 20-24.
- Set 9. Chapter 3: 14, 39a, 40ab, 52, 54, 55a, 56. Due Wednesday, Oct 29.
- Set 10. Chapter 4: 4a, 9, 12; Bonus problems: 8, 10, 15. Due Monday, Nov. 3.
- Set 11. Chapter 4: 17a, 18ab; Bonus problems: 17bcde, 18cd, 21. Due Monday, Nov. 10.
- Set 12. Chapter 4: 22a, 27abc, 32abcd; Bonus: 41. Due Monday, Nov 17.
- Final: We will schedule a time. You will have three hours.

Handouts

- Algebraic Systems
- Properties of the Real Numbers
- Unit Circles in $\mathbb{R}^2$
- Solutions to Hwk Set 1 Not ready!

- Dedekind Cuts Notes
- Derivation of the Trapezoidal Rule Error Estimate
- Review Sheet for Midterm Not ready!
- Bernstein Polynomial Approximations
- Stirling's Formula by Keith Conrad
- Review Sheet for Final Not ready!

Supplemental reading

- Berkeley's Attack on the Infinitesimal, by Mark Monti, a term paper for Math 163 with Prof. Wallach, UCSD, June 2003. Copyright 2004, Regents of the University of California, all rights reserved.
- The Mathematical Shape of Modernity, by Paula E. Findlen, *Chronicle of Higher Education*, June 23, 2014. (Can only be viewed on a University computer or with a subscription.)
- Cantor's Other Proofs that R Is Uncountable, by John Franks, *Mathematics Magazine*, Vol. 83, No. 4 (October 2010), pp. 283-289.
- A pedagogical history of compactness, by Manya Raman-Sundstrom, preprint.
- The emergence of open sets, closed sets, and limit points in analysis and topology, by Gregory H. Moore, *Historia Mathematica*, Volume 35, Issue 3, August 2008, Pages 220–241.
- Zeno's Paradoxes, by Bradley Dowden, in the *Internet Encyclopedia of Philosophy, a Peer-Reviewed Academic Resource.*
- The Continuum Hypothesis, by Peter Koellner, *Stanford Encyclopedia of Philosophy*, May 22, 2013.

Other textbooks on Real Analysis

- *Principles of Mathematical Analysis*, by Walter Rudin. (400 level)
- *Real and Complex Analysis*, by Walter Rudin. (500 level)
- *Understanding Analysis*, by Stephen Abbott. (300 level)
- *The Elements of Real Analysis*, by Robert G. Bartle. (400 level)
- *Real Analysis*, by Halsey Royden. (500 level)
- *Foundations of Analysis*, by Edmund Landau. (Any level)

Books every serious student of mathematics should have

- *The History of the Calculus and Its Conceptual Development*, by Carl B. Boyer Amazon Dover Press B&N
- *Proofs and Refutations: The Logic of Mathematical Discovery*, by Imre Lakatos. See especially Appendices I and II. MAA Amazon

Algebraic Systems

A set G together with a binary operation $+ : G \times G \to G$ is called a **group** provided the following hold.

(1) $(a+b)+c = a+(b+c)$ for all $a, b, \& c$ in G.
(2) There exists $0 \in G$ such that $a+0 = 0+a = a$ for all $a \in G$.
(3) For all $a \in G$ there exists a $-a \in G$ such that $a + -a = 0$.

If in addition $a+b = b+a$ for a and b in G then G is a **commutative** or **abelian group**.

Examples. $(\mathbb{R}, +)$, $(\mathbb{Z}, +)$, $(\mathbb{Z}/n, +)$, $(\mathbb{R} - \{0\}, \cdot)$, $(2 \times 2\,\text{matrices}, +)$, but not $(2 \times 2\,\text{matrices}, \cdot)$, yet $(2 \times 2\,\text{matrices with } \det \neq 0, \cdot)$ is a group.

A set R together with two binary operations $+$ and $\cdot$ from $R \times R$ to R is called a **ring** provided the following hold.

(1) $(R, +)$ is an abelian group.
(2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, \& c$ in R.
(3) $a \cdot (b+c) = a \cdot b + a \cdot c$ for all $a, b, \& c$ in R.
(4) $(a+b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, \& c$ in R.

If there exists an element $1 \in R$ such that $a \cdot 1 = 1 \cdot a = a$ for all $a \in R$ then R is a **ring with a unit**.

If $a \cdot b = b \cdot a$ for all a and b in R then R is a **commutative ring**.

Examples. $(\mathbb{R}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(\mathbb{Z}, +, \cdot)$. $(2 \times 2\,\text{matrices}, +, \cdot)$, { polynomials }, $(\mathbb{Z}/n, +, \cdot)$.

A set F together with two binary operations $+$ and $\cdot$ from $F \times F$ to F is called a **field** provided the following hold.

(1) $(F, +, \cdot)$ is a commutative ring with a unit.
(2) $(F - \{0\}, \cdot)$ is an abelian group.

Examples. $(\mathbb{R}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, but not $(\mathbb{Z}, +, \cdot)$. $(\mathbb{Z}/n, +, \cdot)$ is a field iff n is prime.

A field F is **ordered** provided there exists a order relation $<$ such that that following properties hold for all $x, y, z \in F$.

(1) $x < y \;\&\; y < z \implies x < z$.
(2) One and only one of the following are true: $x < y$, $x = y$, $y < x$.
(3) $x < y \implies x + z < y + z$.

Let $(V, \oplus)$ be an abelian group and let $(F, +, \cdot)$ be a field. Then V is a **vector space** over F if there is a binary operation $\odot : F \times V \to V$, called **scalar multiplication** where the following hold.

(1) $1 \odot v = v$ for all $v \in V$.
(2) $(r \cdot s) \odot v = r \odot (s \odot v)$ for all r and s in F and v in V.
(3) $r \odot (v \oplus w) = (r \odot v) \oplus (r \odot w)$ for all $r \in F$, v and w in V.
(4) $(r+s) \odot v = (r \odot v) \oplus (s \odot v)$

The elements of V are called **vectors**. The elements of F are called **scalars**.