# Ch 2: A Taste of Topology

## Section 1: Metric Spaces

**Def** A metric space is a set $M$ and a function $d: M \times M \to [0, \infty)$ s.t.

1. $d(x,y) = 0$ iff $x = y$. (Positive definite)
2. $d(x,y) = d(y,x)$. (Symmetric)
3. $d(x,z) \le d(x,y) + d(y,z)$ (Triangle ineq.)

**Ex** $\mathbb{R}$ with $d(x,y) = |x-y|$. $\mathbb{Z}$ with $d(x,y) = 1$ if $x \neq y$ and $0$ if $x = y$. (Think about that.)

On $\mathbb{R}^n$ the usual metric is $d((a_1, a_2, \ldots a_n), (b_1, \ldots, b_n)) =$

$$\sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}$$

In a metric space the open ball of radius $r$ and center $p$ is

$$B(p,r) = \{x \in M \mid d(p,x) < r\}.$$

On the next page we plot the "unit circles" in $\mathbb{R}^2$ for several alternative metrics.

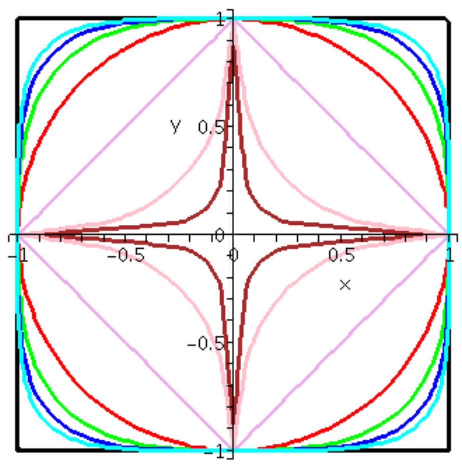
$$d_n((x_1,y_1),(x_2,y_2)) = \sqrt[n]{|x_1 - x_2|^n + |y_1 - y_2|^n} \quad n = 1,2,3,4,5$$

$$d_{\frac{1}{n}}((x_1,y_1),(x_2,y_2)) = \left(\sqrt[n]{|x_1 - x_2|} + \sqrt{|y_1 - y_2|}\right)^n \quad n = 2,3$$

and $d_\infty( \qquad ) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$.

# Unit circles in R^2 using different metrics

```
> with(plots):

> ball1:=implicitplot( abs(x)+abs(y)=1,x=-1..1,y=-1..1,thickness=2,color=plum,numpoints=1000):

> ball2:=implicitplot( x^2+y^2=1,x=-1..1,y=-1..1,thickness=2,color=red):

> ball3:=implicitplot( abs(x^3)+abs(y^3)=1,x=-1..1,y=-1..1,thickness=2,color=green):

> ball4:=implicitplot( x^4+y^4=1,x=-1..1,y=-1..1,thickness=2,color=blue):

> ball5:=implicitplot( abs(x^5)+abs(y^5)=1,x=-1..1,y=-1..1,thickness=2,color=cyan):

> ballhalf:=implicitplot(
  sqrt(abs(x))+sqrt(abs(y))=1,x=-1..1,y=-1..1,thickness=2,color=pink,numpoints=1000):

> ballthird:=implicitplot(
  root(abs(x),3)+root(abs(y),3)=1,x=-1..1,y=-1..1,thickness=2,color=brown,numpoints=1000):

> ballinfinity:=implicitplot(
  max(abs(x),abs(y))=1,x=-1..1,y=-1..1,thickness=2,color=black,numpoints=10000):

> display(ballthird,ballhalf,ball1,ball2,ball3,ball4,ball5,ballinfinity);
```

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>
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**Def** A sequence $(a_n)$ converges to $a$ if

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow d(a_n, a) < \varepsilon$.

**Fact** Limits, when they exist, are unique. That is, if $a_n \to a$ and $a_n \to b$, then $a = b$.

**Pf** Let $\varepsilon > 0$ be given.

Let $N$ be s.t. $n \geq N \Rightarrow d(a_n, a) < \varepsilon/2$.

Let $M$ be s.t. $m \geq M \Rightarrow d(a_m, b) < \varepsilon/2$.

Then, for all $n \geq \max(N, M)$ we have

$d(a,b) \leq d(a, a_n) + d(a_n, b) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Thus, $\forall \varepsilon > 0$, $d(a,b) < \varepsilon$. By the $\varepsilon$-principle (see pg 21), $d(a,b) = 0$. Hence $a = b$. ∎

**Thm** Every subseq of a convergent seq converges to the same limit.

**Pf** See textbook, pg 54.