

Completion

Thm

Every metric space can be completed. This means $\exists$ a complete m. sp. $\hat{M}$ and $h: M \rightarrow \hat{M}$ s.t. $\overline{h(M)} = \hat{M}$ and $h: M \rightarrow h(M)$ is an isometry. (An isometry is a homeomorphism that preserves distances.)

Idea

Take the collection of Cauchy sequences in M and declare two to be equivalent if $d(p_n, q_n) \rightarrow 0$. We can put a "natural" metric on the set of equivalence classes to get $\hat{M}$.

Pf

Let $\bar{p} = (p_n)$ and $\bar{q} = (q_n)$ be Cauchy seq's in M . Define $\bar{p} \sim \bar{q}$ if $d(p_n, q_n) \rightarrow 0$. It is easy to check that $\sim$ is an eq. rel. **(Do This!)**
Let $\hat{M} = \{ \text{Cauchy seq's in } M \} \text{ mod } \sim$.

We define a metric D on $\hat{M}$ by letting

$$D(P, Q) = \lim_{n \rightarrow \infty} d(p_n, q_n)$$

where $(p_n) \in P$ and $(q_n) \in Q$. We need to check the following.

(a) D is a well defined function, meaning that $D(P, Q)$ always exists and does not depend on which seg's are chosen from P and Q .

(b) D is a metric

(c) $\hat{M}$ is complete.

Let $h: M \rightarrow \hat{M}$ be given by $h(p) = [p, p, p, \dots]$.
We must show the following.

(d) h is an isometry

(e) $\overline{h(M)} = \hat{M}$, i.e. $h(M)$ is dense in $\hat{M}$.

Pf of (a) We use Lemma 81 (pg 119) from the textbook:

$$\forall p, q, x, y \in M, |d(p, q) - d(x, y)| \leq d(p, x) + d(q, y).$$

Let $\bar{p}, \bar{p}'$ be seg's in P and $\bar{q}, \bar{q}'$ be seg's in Q .
By Lemma 81

$$|d(p_n, q_n) - d(p_n, q_n)| \leq d(p_n, p_n) + d(q_n, q_n).$$

Therefore $(d(p_n, q_n))$ is a Cauchy seg in $\mathbb{R}$
and hence $\lim_{n \rightarrow \infty} (p_n, q_n)$ exists.

Let $L = \lim d(p_n, q_n)$ and $L' = \lim d(p'_n, q'_n)$.

We claim $L = L'$. This is just the triangle inequality.

$$|L - L'| \leq |L - d(p_n, q_n)| + |d(p_n, q_n) - d(p'_n, q'_n)| + |d(p'_n, q'_n) - L'|$$

The first and third term on the RHS go to zero as $n \rightarrow \infty$. The middle term, by Lemma 81, is

$$\leq d(p_n, p'_n) + d(q_n, q'_n).$$

These two terms go to zero also. Thus $L = L'$, and hence D is independent of the choice of sequences used from the eq. classes P and Q .

Pf of (b) (D is a metric.)

Since the original metric d on M is symmetric, that is,

$$D(P, Q) = \lim d(p_n, q_n) = \lim d(q_n, p_n) = D(Q, P).$$

Clearly,

$$D(P, P) = \lim d(p_n, p_n) = \lim 0 = 0.$$

If $D(P, Q) = 0$, then for $(p_n) \in P$ and $(q_n) \in Q$ $\lim d(p_n, q_n) = 0$. But then $(p_n) \sim (q_n)$ and so $P = Q$. This shows positive definiteness.

Finally, let $p, q, r \in \hat{M}$. We have the ~~tri~~^{tri} ineq:

$$\begin{aligned} D(p, q) &= \lim d(p_n, q_n) \leq \lim d(p_n, r_n) + d(r_n, q_n) \\ &= \lim d(p_n, r_n) + \lim d(r_n, q_n) = D(p, r) + D(r, q). \end{aligned}$$

Thus D is a metric on $\hat{M}$.

Ptd ($\hat{M}$ is complete.) Let (P_k) be a Cauchy seq in $\hat{M}$. We must show it converges in $\hat{M}$. We will construct a candidate for the limit using a diagonalization technique. To do this we'll carefully select $(p_n^k) \in P_k$ for each k .

First selection any $(p_n^k) \in P_k$. $\exists N$ s.t. $n, n \geq N \Rightarrow d(p_n^k, p_n^k) < \frac{1}{k}$. Consider the

~~first~~ subseq obtained by deleting the first N terms: $(p_{n+N}^k)_{n=1}^{\infty}$. It is in P_k . In fact

any subseq of a Cauchy seq will be Cauchy and will be equivalent in our sense to the original sequence. ~~Prove~~ Prove this!

Renumber the seq's $(p_n^k)_{n=N+1}^\infty$ so that they each start from p_1^k . Thus $\forall k \quad (p_n^k)_{n=1}^\infty \in P_k$ and $d(p_a^k, p_b^k) < \frac{1}{k}$ $\forall a, b \geq 1$.

Let $q_n = p_n^1$. Let $Q = [(q_n)]$. We need to check that (q_n) is Cauchy, so that $Q \in \hat{M}$, and that $P_k \rightarrow Q$ in $\hat{M}$.

(q_n) is Cauchy: Let $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $k, l \geq N \Rightarrow D(p_k, p_l) < \frac{\varepsilon}{3}$. Further, we can choose $N > 3/\varepsilon$. Thus $1/k$ and $1/l$ are $< \varepsilon/3$. Now,

$$\begin{aligned} d(q_k, q_l) &= d(p_k^1, p_l^1) \leq d(p_k^1, p_n^1) + d(p_n^1, p_n^l) + d(p_n^l, p_l^l) \\ &< \frac{1}{k} + d(p_n^1, p_n^l) + \frac{1}{l} < d(p_n^1, p_n^l) + \frac{2\varepsilon}{3}. \quad (*) \end{aligned}$$

The limit $\lim_{n \rightarrow \infty} d(p_n^k, p_n^l) = D(p_k, p_l) < \varepsilon/3$. Thus

for n large enough $d(p_n^k, p_n^l) < \varepsilon/3$. Since

the inequality $(*)$ is valid $\forall n$, we have

$$d(q_k, q_l) < \varepsilon.$$

Thus, (q_n) is Cauchy and $Q \in \hat{M}$.

$P_k \rightarrow Q$

Let $\varepsilon > 0$. Choose $N > \frac{2}{\varepsilon}$ s.t. $k, n \geq N \Rightarrow d(q_k, q_n) < \frac{\varepsilon}{2}$.
Then

$$d(p_n^k, q_n) \leq d(p_n^k, p_k^k) + d(p_k^k, q_n)$$

$$= d(p_n^k, p_k^k) + d(q_k, q_n)$$

$$< \frac{1}{k} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Now $\lim_{n \rightarrow \infty} d(p_n^k, q_n) = D(p_k, Q)$. We have

that for large enough k , $D(p_k, Q) < \varepsilon$.

Thus $\lim_{k \rightarrow \infty} p_k = Q$ as required.

Pf of (d)

Recall $h: M \rightarrow \hat{M}$ was given by $h(p) = [(p, p, p, \dots)]$.
If $p \neq q$, then $D((p), (q)) = |p - q| \neq 0$. Thus
 $h(p) \neq h(q)$, so h is one-to-one. This also
shows that h preserves distances.

This fact can be used to show that h , and h^{-1} ,
are continuous. Just use $\varepsilon = \delta$.
on $h(M)$

Pf of (e)

We need to show $h(M)$ is dense in $\hat{M}$. Let $Q \in \hat{M}$.

Let $(q_n) \in Q$. Let $Q_k = h(q_k) = [(q_k, q_k, q_k, \dots)]$.

Then $D(Q, Q_k) = \lim_{n \rightarrow \infty} d(q_n, q_k)$.

Thus, $\lim_{k \rightarrow \infty} D(Q, Q_k) = 0$, since
 (q_n) is Cauchy.

