The Cauchy completion construction of $\mathbb{R}$ from $\mathbb{Q}$

The proof of the Cauchy completion thm does not to completing $\mathbb{Q}$ to get $\mathbb{R}$, because the proof assumed $\mathbb{R}$ was complete. Also, it said nothing about the field operations or the order relation. Now, we do this case, at least in outline.

Let $\hat{\mathbb{Q}}$ be the equivalence classes of Cauchy sequences in $\mathbb{Q}$ as we did before.

Let $P = [(p_n)]$ and $Q = [(q_n)]$ be in $\hat{\mathbb{Q}}$. Define

$$P + Q = [(p_n + q_n)] \,, \quad PQ = [(p_n \cdot q_n)]$$

$$P - Q = [(p_n - q_n)] \,, \quad P/Q = [(p_n/q_n)],$$

where for the last $Q \neq [(0,0,0,\dots)]$ and the $(q_n) \in Q$ choosen cannot have any 0 entries.

P is positive if $\exists\ (p_n) \in P$ s.t. $\exists\, \alpha > 0$ ~~s.t.~~ $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow p_n > \alpha$. (Thus "limit p_n" > 0.)

Define $P < Q$ if $Q - P$ is positivea.

We identify $r \in \mathbb{Q}$ with $[(r,r,r,\dots)] \in \hat{\mathbb{Q}}$.

Exercise 134: Show that $\hat{\mathbb{Q}}$ is a field.

Exercise 135: Show that $<$ is an order on $\hat{\mathbb{Q}}$ ~~with~~ that extends the order on $\mathbb{Q}$.

Claim: $\hat{\mathbb{Q}}$ has the lub property.

Pf: Let $\mathcal{P} \subset \hat{\mathbb{Q}}$ and suppose $B \in \hat{\mathbb{Q}}$ is an upper bound for $\mathcal{P}$, that is $\forall P \in \mathcal{P}$, $P \leq B$.

Fact 1: Let $P = [(p_n)]$ and $Q = [(q_n)]$. Then $P < Q$ iff $\exists \alpha > 0, N \in \mathbb{N}$ s.t. $m,n \geq N \Rightarrow p_n + \alpha < q_m$.

Pf: ($\Rightarrow$) Since $Q - P$ is positive $\exists \alpha > 0, N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow q_n - p_n > 2\alpha$. Thus $p_n + 2\alpha < q_n$.

Since (q_n) is Cauchy $\exists M$ s.t. $k, l \geq M \Rightarrow |q_k - q_l| < \alpha$, or $-\alpha < q_k - q_l < \alpha$. Thus $q_k < \alpha + q_l$. Thus, for $m,n \geq \max\{N, M\}$ we have

$$p_n + 2\alpha < \alpha + q_m \quad \text{or} \quad p_n + \alpha < q_m.$$

($\Leftarrow$) This ~~follows~~ by definition. $\square$

Fact 2 We can choose $(b_n) \in B$ s.t. all the terms are within 1 of each other.

Pf Let $(b_n)_{n=1}^{\infty} \in B$. $\exists N$ s.t. $m, n \geq N \Rightarrow |b_n - b_m| < 1$. Now $(b_n)_{n=N}^{\infty}$ is also in B, so we can use it. $\square$

Let $b = b_1 + 7$. Then $[(b, b, b \ldots)] > B$ and thus is also an upper bound for $\mathcal{P}$. Let $m \in \mathbb{Z}$ s.t. $m \geq b$. Then $[(m, m, m, m, m, \ldots)]$ is an upper bound for $\mathcal{P}$.

Let q_0 be the smallest integer s.t. $[\overline{q_0}]$ $= (q_0, q_0, q_0 \ldots)$ is an upper bound for $\mathcal{P}$. (q_0 exists since $\mathcal{P} \neq \emptyset$.)

Let q_1 be the smallest multiple of $\frac{1}{2}$ s.t. $[\overline{q_1}]$ is an u.b. for $\mathcal{P}$. Notice $q_1 = q_0$ or $q_0 - \frac{1}{2}$.

Let q_2 be the smallest multiple of $\frac{1}{4}$ s.t. $[\overline{q_2}]$ is an u.b. for $\mathcal{P}$. $q_2 = q_1$ or $q_1 - \frac{1}{4}$.

Let q_3 be the smallest multiple of $\frac{1}{8}$ s.t $[\overline{q_3}]$ is an u.b. for $\mathcal{P}$. $q_3 = q_2$ or $q_2 - \frac{1}{8}$.

In general, Let q_n be the smallest multiple of $\frac{1}{2^n}$ s.t. $[\overline{q_n}]$ is an u.b. for $\mathcal{P}$. $q_n = q_{n-1}$ or $q_{n-1} - \frac{1}{2^n}$.

Let $Q = [(q_n)_{n=0}^{\infty}]$.

(q_n) is monotone non increasing and it is easy to show it is Cauchy (see textbook, p 123).
Thus, $Q \in \hat{\mathbb{Q}}$.

* Suppose Q is not an u.b. for $\mathcal{P}$. Let $[(p_n)] = P \in \mathcal{P}$ be s.t. $Q < P$. By Fact 1 $\exists\, \alpha > 0$, $N \in \mathbb{N}$ s.t.

$$m, n \geq N \implies q_m + \alpha < p_n.$$

Thus, using $m = N$, we have $[\overline{q_N}] < P$. But $[\overline{q_N}]$ is an u.b. for $\mathcal{P}$ by construction!
Thus Q must be an u.b. for $\mathcal{P}$.

* Suppose $R = [(r_n)] \in \hat{\mathbb{Q}}$ is an u.b. for $\mathcal{P}$ and $R < Q$. Then $\exists\, \alpha > 0$ $N \in \mathbb{N}$ s.t.

$$m, n \geq N \implies r_m + \alpha < q_n.$$

Choose $K \geq N$ s.t $\frac{1}{2^K} < \alpha$. Thus, $\forall\, m \geq N$

$$r_m < q_K - \alpha < q_K - \frac{1}{2^K}$$

Thus $R < [\overline{q_K - \frac{1}{2^K}}]$. Since R is an u.b. for $\mathcal{P}$ so is $[\overline{q_K - \frac{1}{2^K}}]$ which contradicts the definition of q_K. Thus Q is the l.u.b of $\mathcal{P}$ $\blacksquare$