2 Continuity

Def Let (M, d) and (N, d') be metric spaces. Let $f: M \to N$ be a function. Then f is <u>continuous</u> at $x \in M$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon.$$

If f is cont. at each $x \in M$, then we say f is cont. on M.

Thm $f: M \to N$ is cont. on M iff $a_n \to a \Rightarrow f(a_n) \to f(a)$.

Pf Let f be cont at a and $a_n \to a$. Let $\varepsilon > 0$. Let $\delta > 0$ be s.t. $d(a, x) < \delta \Rightarrow d'(f(a), f(x)) < \varepsilon$. $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow d(a, a_n) < \delta$. Thus, $n \geq N \Rightarrow d'(f(a), f(a_n)) < \varepsilon$. Hence, $f(a_n) \to f(a)$.

Let $a \in M$ and suppose $a_n \to a \Rightarrow f(a_n) \to f(a)$. We will suppose f is not cont. at a and deduce a contradiction. $\exists \varepsilon_0 > 0$ s.t. $\forall \delta > 0, \exists x \in M$ with

$$d(a, x) < \delta \text{ but } d(f(a), f(\cancel{x_n})) \geq \varepsilon_0.$$

Thus, for each $n \in \mathbb{N}$, $\exists a_n \in B(a, \frac{1}{n})$ s.t. $d(f(a), f(a_n)) \geq \varepsilon_0$. Thus $a_n \to a$, but $f(a_n) \not\to f(a)$. ∎

Corollary If $f: M \to N$ and $g: N \to P$ are cont. then so is $g \circ f: M \to P$.

Pf: See textbook.

Def Let $f: M \to N$ be cont., one-to-one, onto, with $f^{-1}: N \to M$ cont. Then we say f is a homeomorphism and M and N are homeomorphic. This is an equivalence relation on the class of metric spaces.

Ex $(0,1) \approx (0,2)$, $(0,1) \approx \mathbb{R}$. $[0,1] \approx [0,2]$. $[0,1) \approx (0,3]$. These are homeomorphic pairs. But $[0,1]$, $(0,1)$ and $[0,1)$ are not homeo as we will see later.

Note The set of all homeomorphisms from a metric space M to itself forms a group under composition.